The Pacific Institute for the Mathematical Sciences

π in the Sky

June 2001

We've Got Your Number

Sensitive Questions

Constructing Fractals

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This journal is devoted to cultivating mathematical reasoning and problem-solving skills and preparing students to face the challenges of the high-technology era.

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Cover Page: The picture on the cover page was taken at Old Scona Academic High School in Edmonton. Old Scona was established in 1976 to provide motivated students who have demonstrated success and potential for growth with an opportunity to pursue studies that challenge and enrich their learning experience. The school offers the International Baccalaureate program in English, History, and Biology, with instructional focus on university preparation.

If you would like to see your school on the cover page of *π in the Sky*, please invite us for a short visit to meet your students and staff.

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FREE FORUM*

This column is an open forum. We welcome opinions on all mathematical issues: research; education; and communication. Please feel free to write us.

Opinions expressed in this forum do not necessarily reflect those of the editorial board, PIMS, or its sponsors.

Perils of Modern Math Education

by Stuart Wachowicz†

> *"Pride in craftsmanship obligates the mathematicians of one generation to dispose of the unfinished business of their predecessors."*
> -*E.T. Bell, The Last Problem*

The above statement most accurately describes the legacy of one generation of mathematicians to the next. However, on might be tempted to ponder whether this will continue to be possible in North America. The discipline of mathematics, as we have known it, is clearly under threat. The threat is a consequence of allowing curriculum writers to change the centuries-old definition of mathematics and what needs to be learned based on utilitarianism, combined with the current practice of allowing unproven fads to influence pedagogy.

A century ago the utilitarian threat was expressed curtly by Andrew Carnegie, who stated, *"Schools are a place where children learn how to manufacture!"* Today, the same idea is masked in the notion that mathematics, to be of value, must be studied in a way that always gives a "real world" (whatever that means) application. While it is true that students benefit when they see the power of mathematics at work in deriving a solution to a common situation, there is also the broader aspect of the discipline of mathematics, one that empowers the individual to appreciate this most elegant and exact language. I am reminded of a statement written by Harold Jacobs in a forward to his book, *Mathematics: A Human Endeavour,*

> *"Some of the topics in this book may seem of little practical use, but the significance of mathematics does not rest on its practical value. It is hard to believe that someone flying over the Grand Canyon for the first time would remark. "What good is it?" Some people say the very same thing about mathematics. A great mathematician of our century, G.H. Hardy, said, "A mathematician, like a painter or a poet, is a maker of patterns." Some of these patterns have immediate and obvious applications; others may never be of any use at all. But, like the Grand Canyon, mathematics has its own beauty and appeal to those who are willing to look."*

In a quest for utilitarian value, much of the modern mathematics curriculum in the public school system no longer seriously attempts to inculcate a deep understanding of what Newton referred to as "*the language of the universe*." Lip service is paid to the goal of helping students to become more adept at problem solving, but modern curricula fail to place the emphasis on the foundation of problem solving—the mastery of number relationships and the fundamental axioms and postulates upon which mathematical reasoning is based. Virtually all students can and should learn these.

The second threat is that of allowing politically expedient fads to unduly influence public school pedagogy. If many centuries ago the developers of the abacus were able to market to society the concept that this new technology could remove from students the need to become masters of basic arithmetic calculation, we may have seen mathematics take a different turn. Certainly the argument may have been valid for those societies utilizing very cumbersome number systems, such as in Greece and Rome, but even in areas that adopted Hindu numeration, the abacus did not remove the perception that mastery of hand calculation was still needed by the person who was even partially educated. Even with later technological innovation, no one seriously considered that mastery of hand calculation was no longer necessary. Today, however, given advances in microelectronics (a consequence of traditional rigor in mathematics and science), there are those who promote the replacement of mastery of numerical operations with the use of a calculator. Throughout the U.S. and Canada there exist "educators" who are always seeking something new and innovative. This is seldom connected with quantified research to determine if the innovation actually produces a better result, but it can generate a graduate degree and place one on the lucrative speaking circuit. Driven by progressivist ideology, they seek to liberate students from the drudgery of calculation, especially the dreaded long division.

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Given that calculators of the mid 1970s were incapable of handling fractions, curricula were altered to allow decimals to be introduced earlier, pushing aside the antiquated fraction. The role of fractional operations was thus reduced in elementary school, and it was left to the junior high teacher to address. Alas, calculators grew in intellect and soon learned how to work with fractions. Modern curriculum documents (such as the Western Canadian Protocol Framework for Mathematics) began to include instructions that students would do such calculations with pencil and paper **and/or** with a calculator. As a consequence, many a teacher, parent, and employer now lament the fact that graduates display an inability to comprehend or work with rational numbers. High school teachers dealing with students who have not internalized number relationships, previously ingrained as a result of mastery of paper and pencil calculation, experience increasing difficulty in developing fluency with rational algebraic expressions.

Only a few years ago, calculator manufacturers introduced the graphing calculator. The gurus immediately lobbied for more curriculum change to allow this new innovation to take math students (already lacking a numerate foundation) to a "new" level of understanding. No longer would they have to labour to calculate the parameters of a hyperbola. Just punch in the coefficients and watch the little lines move on the screen. No longer would students have to become proficient at completing the square of quadratics, or memorizing the unit circle. Instant recall from the mind could be replaced with a microchip. Those making these decisions never stopped to ask who was using graphing calculators beyond high school. The fact that there is virtually no application seemed to be missed. The fact that universities do not permit these instruments on examinations was not considered. Even the concept of memorization, the greatest tool for developing mental capacity, was spurned.

There is a logical fallacy at work here that few in public education seem to be willing to expose. New approaches to public school math postulate that without an ingrained knowledge base of number relationships and without fluency in calculation, symbolic manipulation and formal training in reasoning, students are still able to grasp algebraic, trigonometric, and geometric principles at a level that will enable them to become effective problem solvers in mathematics. The fact that fewer than 10% of the mathematics graduate students in our province received their early education in North American public schools may cast some doubt on this theory, which disregards the collective wisdom of centuries. The truth is that without an appreciation for the discipline of mathematics, developed from an early age, mathematical reasoning and potential is impeded.

The new approach embraces the notion that technology is inexorably linked with the discipline of mathematics. Technology is but a consequence of mathematics. Real mathematics is in fact an independent form of technology. Today, however, technology is mindlessly driving educational philosophy, curriculum design and assessment. Mathematical reasoning is thus impeded and held hostage to this anti-intellectual, technological imperative.

While technology may have many positive applications in education such as helping an instructor amplify a concept, its current overuse is a problem from both a financial and a pedagogical perspective. The imperative implies that because there is so much information, it is impossible to know it all. Therefore, instead of students becoming knowledge rich, they must become skilled at accessing information. This too is a logical fallacy. One may go so far as to postulate the present imperative constitutes a war on both memory skills and the establishment of a broad knowledge base. Knowledge has been, is, and always will be the raw material of reason, and without an internal knowledge base, process skills become ineffectual. Nowhere is this more true than in mathematics.

If we are to have the craftsmen to dispose of the unfinished business of our predecessors, as Bell observed, students in public schools must be given the knowledge and skills that will enable them to do just that. When the Emperor is naked there is a responsibility for those aware of his condition to have the courage to so inform him.

Enjoy symbolic Algebra with a new calculator!

Math Investigations

We've Got Your Number

Ted Lewis[†]

Your life is filled with code numbers. Every commercial product has a 12-digit number called the UPC (the Universal Product Code). The UPC number is written as a barcode so it can be read by scanners and in decimal form so it can be read by humans. Soon, all products will carry a 13-digit barcode number that looks a lot like a UPC—in fact, it is a superset of the UPC called the EAN (European Article Number). To order a book, you may have to supply its ISBN (International Standard Book Number). To subscribe to a magazine, you may be asked for its ISSN (International Standard Serial Number).

Open your wallet and check your student ID card. It likely has a code number on it. Your driver's permit has a 'license number' and it is probably accompanied by a barcode or a magnetic strip as well.

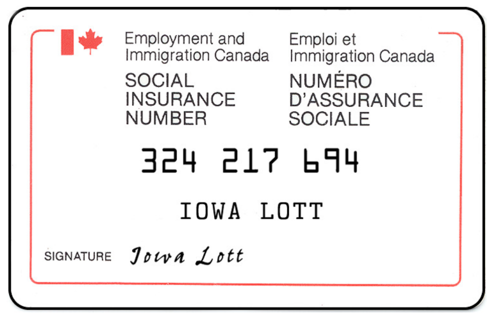
Your credit card has a 16 digit code on it. If you order something over the internet, you will be asked to provide that code. If you make a mistake entering the digits, you will see a message like, "*Invalid VISA number! Please check the number and re-enter.*"

A few years ago, I encountered a similar situation when I was using a computer program to prepare my income tax return. I wanted to see what the tax would be for variety of taxpayers, so I made up some fictitious data for "Richard Richman." As part of the data, I included my own Social Insurance Number (S.I.N.). The program promptly rejected this because my S.I.N. was already in the small database it was creating on my hard drive, and two people cannot have the same number. So, I made up a completely arbitrary one, but that didn't get me very far—the program told me that I had entered an invalid number, and it wouldn't let me continue until I provided an acceptable one.

How can an on-line book company tell when you have entered an incorrect VISA number? How did the income tax program know that I was entering a fake S.I.N.? This "magic" is accomplished by using what is called an *error-detecting code* and, as is true of all magic, the idea behind error detection is quite simple.*

The IBM scheme

For validation, most error-detecting schemes use a *check digit*. This is usually the rightmost digit of the code. The other digits, the *information digits*, can be freely chosen, but the check digit is calculated. For Canadian S.I.N.s and for many credit cards, the check digit is computed using a method devised by IBM. Spaces have no significance and are only there to make it easier to read the number.



Here is how the IBM scheme is used to validate Iowa Lott's S.I.N. Beginning with the rightmost check digit, identify the alternate digits. I have put them in boxes.

$$\boxed{3}\,2\,\boxed{4}\,2\,\boxed{1}\,7\,\boxed{6}\,9\,\boxed{4}$$

Add the boxed digits: $3+4+1+6+4=18.$

Multiply the other digits by 2: $4,\ 4,\ 14,\ 18.$

Add the *digits* of these numbers:

$$4+4+1+4+1+8=22.$$

Add the two results: $18+22=40.$

(Note that in the third step we do not add the numbers; rather, we add the *digits* of the numbers.) The S.I.N. is considered to be valid if the result is divisible by 10, and so Iowa Lott's number passes the validation test.

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*When you listen to a CD, the music is brought to you courtesy of an *error-correcting code*. That code not only detects errors, it also repairs them. The digitally encoded music on a CD has such strong error-correcting capabilities that apparently you can drill a 2.5 mm hole through your CD and it will still play flawlessly. PIMS does **not** advocate that you try this!)

Calculating the check digit

The validation procedure tells us how the check digit is found: carry out the calculations with x in the place of the check digit and solve for x. For example, suppose the information digits for your S.I.N. are 22501008. Then your S.I.N. will be 225-010-08x, and x is calculated as follows:

Identify the alternate digits:

$$\boxed{2}\,2\,\boxed{5}\,0\,\boxed{1}\,0\,\boxed{0}\,8\,\boxed{x}$$

Add the boxed digits: $2+5+1+0+x = 8+x$.
Multiply the other digits by 2: $4,\ 0,\ 0,\ 16$.
Add the *digits* of these numbers: $4+1+6 = 11$.
Add the two results: $8+x+11 = 19+x$.

To make $19+x$ divisible by 10, the digit x must be 1, and the social insurance number would become 225-010-081.

How good is the error detection?

The most common errors in entering numbers are reported to be:

- entering one of the digits incorrectly; or
- interchanging two adjacent digits.

No error-detection scheme can flag all errors, and so they are designed to catch only the most common ones. At the very least, an error-detection scheme should flag either of the above.

> The IBM method will detect an error if a single digit is changed. This includes the case where the check digit is changed.

To illustrate why, let us see what happens if the digit 7 on Iowa Lott's credit card is changed to something else (see the picture on page 5). That is, suppose that instead of entering 7, you enter an x, and this is the only error that you make.

The credit card number is

$$4\,\boxed{0}\,0\,\boxed{2}\,1\,\boxed{2}\,6\,\boxed{5}\,x\,\boxed{0}\,2\,\boxed{1}\,0\,\boxed{6}\,9\,\boxed{3}.$$

The position of the digit x means that it is one of the digits that will be multiplied by 2 during the validation process. Depending upon x, the number $2x$ could be either a single digit or a double digit number. We consider each case separately.

If $0 \le x < 5$ (so $2x$ is a single digit).
Carrying out the IBM validation procedure, the final sum will be $45 + 2x$ (try it). No matter what the digit x is, this will not be divisible by 10, and so an error will be detected.

If $5 \le x \le 9$ (so $2x$ is a two-digit number).
The digits of $2x$ will be 1 and $2x - 10$. Carry out the validation and you will get a final sum of $36 + 2x$. Since $x \ne 7$ and since $5 \le x \le 9$, this sum will also fail to be divisible by 10, and an error will be detected.

The IBM method is very good at detecting an error if a single digit is entered incorrectly. Although it is not completely successful in detecting adjacent switches, it will detect most of those errors as well.

> The IBM method will detect an error if adjacent digits are interchanged, *provided the two digits are not 9 and 0.*

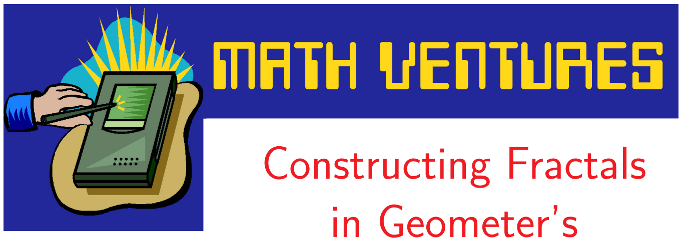
We leave it to the reader to verify the statement.

The IBM method is not the only one that is used for error detection. UPC and EAN numbers use a similar scheme. (For both codes, alternate numbers are tripled instead of doubled, and the numbers, not the digits, are added. Numbers are considered valid if they are divisible by 10). The ISBN error-detection method is based on divisibility by 11. It will detect an error if a single digit is changed, or if two digits are swapped, even if the digits are not adjacent.

Error correction requires more check digits than error detection. On page 5, it was mentioned that CDs have very strong error-correction capabilities. This comes at a price—most of the binary digits on a CD are check digits. Only about one-third of the data is music. The stronger the error correction, the more redundancy (check digits) is required. Conversely, if there is lot of redundancy in the data, that is a signal there might be error correction going on. Do you not find it interesting that geneticists working on the human genome project have been quoted as saying there seems to be a lot of extra junk and redundancy in our DNA? In the real world, mathematics can be found everywhere.

Math Jokes

You might be a mathematician if...
You are fascinated by equations.
You know by heart the first fifty digits of π.
You have tried to prove Fermat's Last Theorem.
You know ten ways to prove the Pythagorean Theorem.
Your telephone number is the sum of two prime numbers.



Constructing Fractals in Geometer's SketchPad

Michael P. Lamoureux†

The new Western Canada Protocol for Mathematics requires high school students to be familiar with fractals, which are a type of geometric object with self-similarity and recursive properties. Many school teachers and their students already use software tools to demonstrate and explore geometric concepts on the computer. We describe how to build fractals using the familiar Geometer's SketchPad^{TM} software.

Fractals and Geometer's SketchPad

A fractal is a geometric shape that has a basic property of self-symmetry: roughly speaking, parts of the shape look like small copies of the whole. It is a strange notion when you first hear of it, but when you see a few examples the concept becomes clear. One well-known example is the Sierpinski gasket, a very beautiful simple fractal, shown in Figure 1 below.

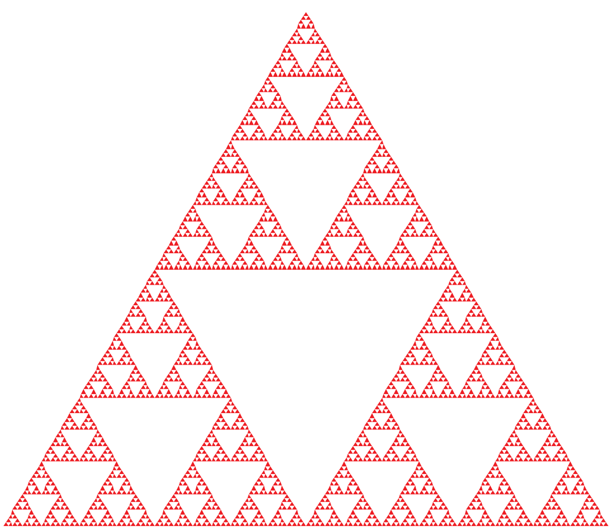


Figure 1: The Sierpinski Gasket

You can quickly see large triangles in the shape, with repetitions of smaller triangles inside. It is not hard to notice the top half of the gasket is an exact copy of the whole thing, at one-half the size. Indeed, the gasket is repeated three times in itself, once at each corner, each exactly one-half the size of the whole.

Another example, shown in Figure 2, is called the Koch Curve. Here the self-similarity may not be immediately obvious, but notice the basic shape of the whole curve as one large bump surrounded by two smaller bumps. This pattern is repeated throughout the curve, at various smaller sizes. It shouldn't take long before you notice the whole curve is really four copies of itself, at exactly one-third the size.

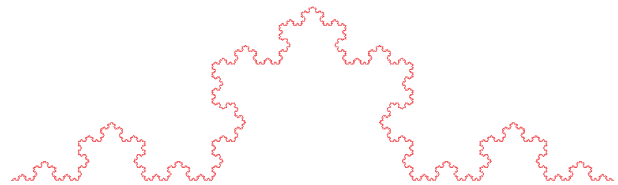


Figure 2: The Koch Curve

Elegant as these figures are, they can be a challenge to draw. Their fractal properties make them well suited to construction on a computer, but students can quickly become mired in programming details if they try to build a computer program to create these forms. A solution is to use a computerized drawing package that has all the necessary commands to build a fractal from scratch. Geometer's SketchPad is just such a package.

Geometer's SketchPad (or GSP) is a handy tool widely used in high schools and colleges for exercises in geometry and for explorations of geometrical constructions. This software is akin to a "word processor" for geometry, including such basic objects as points, lines, and circles, and provides a variety of point-and-click tools to manipulate those objects in geometrically useful ways. The software "knows" how to do many standard straight-edge and compass operations, transformations, and constructions.

What GSP also provides is a simple scripting tool that allows a student to record a series of geometric constructions, then repeat them over and over again. This repetition, or iteration, of basic commands becomes the tool for building self-similar fractals.

This article provides a brief tutorial on how to create fractals in Geometer's SketchPad. I'll assume the reader is familiar with the basic operations of this software—even if you're not, an hour's review should be enough to become comfortable with the basics. A number of fractals will be constructed using a form of iteration as a basic construction step. In each case, the "looping" instruction in a GSP script is used repeat some sequence of elementary operations and create the fractal. We will begin with a simple circle construction to demonstrate how scripting and iterations work, and progress to more and more

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complex examples. This will provide a basis for further explorations on your own.

A key step in every fractal construction is the doubling (or tripling, or more) occurring in each recursion. This leads to exponential growth of the number of geometric elements on the screen—so don't iterate too deeply or your computer may have difficulties. It also leads to the interesting properties (visually and otherwise) of fractal constructions. Once the fractal is built, it is an entertaining challenge to try to measure geometric properties of the resulting shape: length of the perimeter, area, or number of lines/circles/points in the fractal. But this is an exercise for another day.

A Simple Fractal With Circles

Since GSP uses circles as one of its basic constructions, it is not surprising that one of the easiest fractals to construct is a nest of circles, as shown in Figure 3.

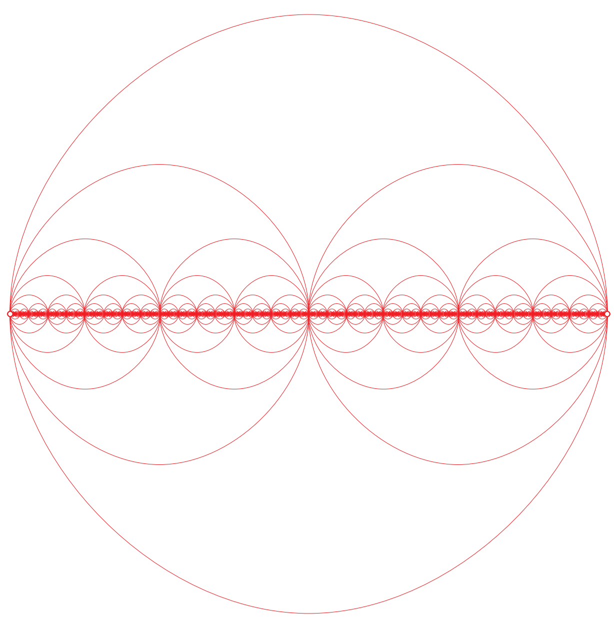


Figure 3: A Circle-based Fractal

The basic iteration begins with a circle in which two smaller circles are drawn. To achieve this geometrically, it is convenient to base the first circle on two points, and then base all smaller circles on some constructed pairs of points. The circles are drawn with center at the midpoint of a line segment connecting the pair of points. The fractal iteration is achieved by repeating this circle construction, first on the left endpoint and midpoint of the segment, then on the right endpoint and midpoint of the segment. The first few steps of the iteration are shown in Figure 4.

To build the GSP script that accomplishes this, begin with a pair of points. The sequence of steps will be as follows: join two points with a segment, then draw a circle with this segment for its diameter. Use the midpoint and endpoints of the segment for the iterations.

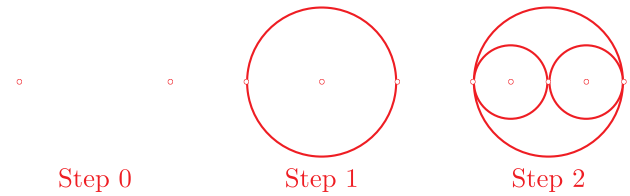


Figure 4: Basic Circle Iteration

We create a script first by opening a new "Sketch" in GSP, then opening a new "Script." Each of these two commands are found under the "File" menu, and each will open its own window.

In the "Script" window, click on the "Record" button to begin transcribing the graphic operations that will be done in the "Sketch" window.

Now the constructions. Switching to the "Sketch" window, create two points on the screen using the "Point" tool. Draw a segment to connect the two points, either using the "Segment" tool, or using the command under the "Construct" menu. Under the "Construct" menu, create a midpoint for the segment. Now draw the first circle using the circle tool, with the center at this new midpoint, and the width set to span the segment. (The circle tool will click automatically to the correct size as you drag the mouse towards the segment's endpoint.)

Now for the iterations. Shift-click to select an endpoint and the midpoint; on the "Script" window, click on the "Loop" button, to tell the script to iterate on these two points. Then Shift-click to select the other endpoint and the midpoint, and again click "Loop" to set another iteration.

Finally, to clean up the picture, click on the line segment and hide it with the "Hide Line" command on the "Display" menu. Then click on the midpoint and hide it as well.

The script is done. Click on the "Stop" button in the "Script" menu, and the script is ready to run. First, clear the "Sketch" window, put on two new points, select them, then click on the "Play" button to start the script. The computer will ask you how many iterations you want to run; try just 1 iteration the first time, to see that the script works as expected. Try again with 5 or 10 iterations, to see the fractal form.

```
Given:
  1.  Point A
  2.  Point B
Steps:
  1.  Let [j] = Segment between Point A and Point B (hidden).
  2.  Let [C] = Midpoint of Segment [j] (hidden).
  3.  Let [c1] = Circle with center at Midpoint [C] passing
      through Point A.
  4.  Recurse on [C] and A.
  5.  Recurse on [C] and B.
```

Figure 5: The Circle Script

If this is not working for you, look over the script recorded in Figure 5. Notice that it is only five lines long, and uses only two points as initial data. Your script should look something like the one in the figure. Be sure to have only two points selected when you "Play" the script. Unfortunately, there is no way to edit a script once you have recorded your actions. It is best to start from scratch if you are having problems.

Once the fractal is made, try dragging around the initial two points—the whole fractal will follow them around. This is part of the power and attraction of using GSP in fractal studies.

A Four-Circle Fractal

The last example with circles can be extended by setting four smaller circles inside each large circle. The resulting fractal gives a wonderful geometric figure reminiscent of a Pysynka, or Ukrainian Easter egg, as shown in Figure 6.

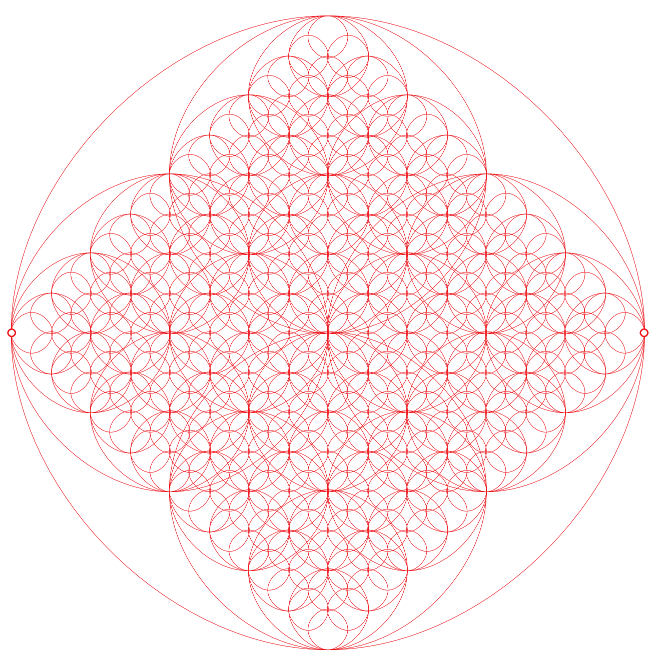


Figure 6: Four-Circle Fractal

The basic iteration is shown in Figure 7, where the initial circle is filled with four overlapping, smaller circles at right angles to each other.

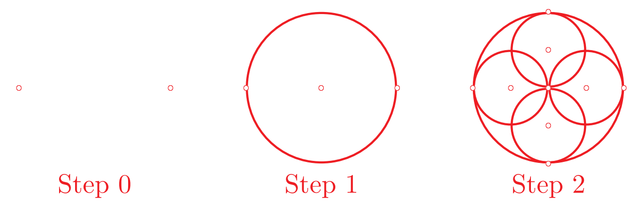


Figure 7: Four-Circle Iterations

Again, the circles and resulting fractal are based on an initial selection of two points. The only new feature used here is GSP's construction tool which makes a perpendicular line to the circle's diameter. This is then used to construct the third and fourth inside circles.

As before, open a new "Sketch" in GSP, and open a new "Script." Click on "Record" to begin the creation of the script.

Keeping in mind the steps are being recorded, create two new points and draw a segment to connect them. Under the "Construct" menu, create a midpoint for the segment. Draw the first circle with the circle tool, with center at this new midpoint, and width set to span the segment. (Again, the circle tool will click automatically to the segment endpoint as you drag towards it. You may prefer to use the menu command that draws a circle automatically from two points.)

```
Given:
  1.  Point A
  2.  Point B
Steps:
  1.  Let [j] = Segment between Point A and Point B (hidden).
  2.  Let [C] = Midpoint of Segment [j] (hidden).
  3.  Let [k] = Perpendicular to Segment [j] through
      Midpoint [C] (hidden).
  4.  Let [c1] = Circle with center at Midpoint [C] passing
      through Point A.
  5.  Let [D] = Intersection of Circle [c1] and Line [k] (hidden).
  6.  Let [E] = Intersection of Circle [c1] and Line [k] (hidden).
  7.  Recurse on [C] and A.
  8.  Recurse on [C] and B.
  9.  Recurse on [C] and [D].
  10. Recurse on [C] and [E].
```

Figure 8: Four–Circle Script

Select the segment again, and the midpoint, then build a perpendicular line by selecting "Perpendicular Line" under the "Construct" menu. Now select the perpendicular line and the circle, then choose "Point at Intersection" under "Construct" to create the two points of intersection of the circle and line.

Now the iterations. There are four pairs of points on which the script will iterate. Shift-click to select an endpoint of the initial segment and its midpoint; on the "Script" window, click on the "Loop" button, to tell the script to iterate on this pair of points. Then Shift-click to select the other endpoint and the midpoint, and again click "Loop" to set another iteration. Then repeat this for the midpoint and one intersection point of the line and circle; then the fourth iteration using the other intersection point.

To clean up the picture, click on the line segment and hide it with the "Hide Line" command on the "Display" menu. Then click on the midpoint, the perpendicular line, and its two intersection points, and hide them all as well.

The script is done. Click on the "Stop" button in the "Script" menu, and the script is ready to run. First, clear the "Sketch" window, and put on two points, select them, then click on the "Play" button to start the script. The computer will ask you how many iterations you want to run; try just 1 iteration the first time, to see that the script works as expected. Try again with 5 or 10 iterations, to see the fractal form.

The resulting script is only a little more complex than the first example. Figure 8 gives an example from a test recording.

The Broccoli Fractal

Figure 9 shows a sample of the well-known broccoli fractal, so-called because of its similarity to a head of real broccoli. Notice the branching bushes of polygons—this is a useful fractal for demonstrating to students methods for computing areas, perimeters, and dimensions of iterated geometric figures.

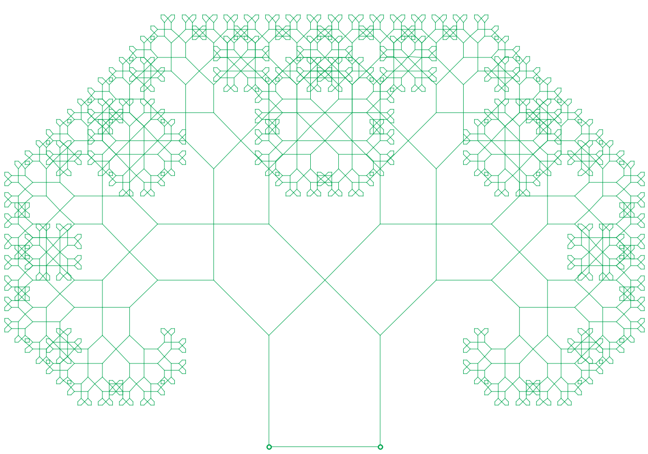


Figure 9: Broccoli Fractal

The basic iterated figure is a five-sided polygon, essentially a square with a right-angled roof perched on top. Two smaller copies of the polygon get attached to the short sides of the top, as shown in Figure 10. Creating the script for this figure is somewhat more complicated, because making a square takes several steps in GSP, as does making triangles.

To record the script, make two initial points and join them with a horizontal segment. Rotate the segment and its endpoint by 90 degrees around the other endpoint, to obtain one vertical side of the square. Reverse endpoints to get the other side of the square. Rotate the sides up by 135 degrees to get the right triangle on top, with extensions past the top vertex of the triangle.

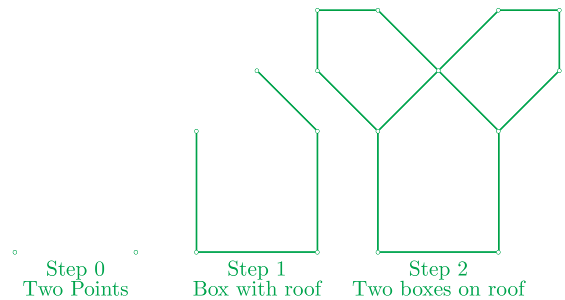


Figure 10: Broccoli Iteration

With the top slanted lines selected, make an intersection point with the "Construct" menu. Hide the too-long segments in the triangle, and replace with segments of the proper size.

To set the iteration steps, select the two points on one of the triangle's top legs (order of selection of the points is important), and click "Loop." Then select the endpoints of the other leg, and click "Loop" again. Finally, hide any extra points or lines that were created. The script is done.

A sample script is shown in Figure 11.

```
Given:
  1.  Point A
  2.  Point B
Steps:
  1.  Let [j] = Segment between Point A and Point B.
  2.  Let [B'] = Image of Point B rotated 90 degrees
      about center Point A (hidden).
  3.  Let [j'] = Image of Segment [j] rotated 90 degrees
      about center Point A.
  4.  Let [A'] = Image of Point A rotated -90 degrees
      about center Point B (hidden).
  5.  Let [j'] = Image of Segment [j] rotated -90 degrees
      about center Point B.
  6.  Let [j''] = Image of Segment [j'] rotated -135 degrees
      about center Point [A'] (hidden).
  7.  Let [j''] = Image of Segment [j'] rotated 135 degrees
      about center Point [B'] (hidden).
  8.  Let [C] = Intersection of Segment [j''] and
      Segment [j''] (hidden).
  9.  Let [k] = Segment between Point [C] and Point [B'].
  10. Let [l] = Segment between Point [C] and Point [A'].
  11. Recurse on [B'] and [C].
  12. Recurse on [C] and [A'].
```

Figure 11: Broccoli Script

The Tree

Trees are one of the most basic fractal shapes: the form of a tree starts with a main trunk, the trunk splits into a number of branches, the branches extend and split into sub-branches, and so on. Surprisingly, this can be a tricky fractal to create in GSP, because of the variety of transformations needed to create sub-branches from the original trunk: translations by vectors; rotations; even dilations can be used.

Figure 12 shows a simple tree: each branch splits into two at the joints, and each sub-branch is the same length as the original. For simplicity, we avoid dilations so that each branch is the same length as the trunk.

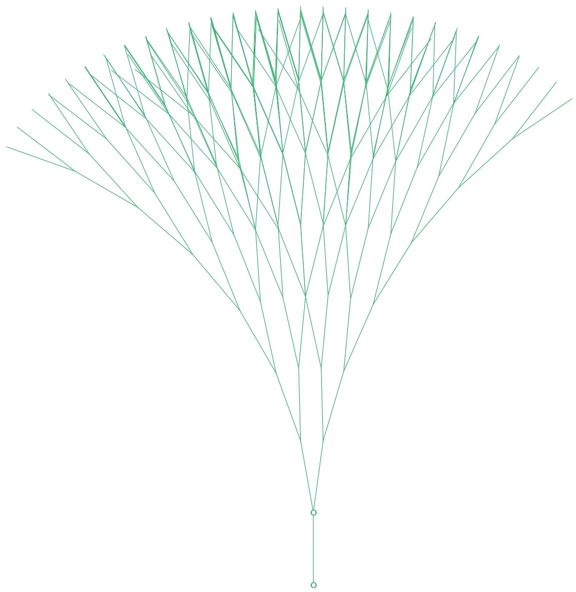


Figure 12: Simple Tree

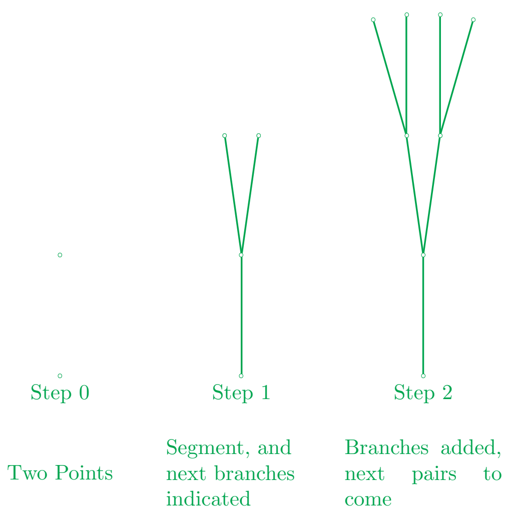


Figure 13: Simple Tree Iteration

Two repeats of the basic iteration are shown in Figure 13. Note in the first iteration, only one segment is drawn, and two new endpoints for the branches are created. The branches themselves don't get drawn until the next loop of the iteration.

The steps of the construction are as follows. After beginning the recording, create a vertical line segment from two points, then translate the top point vertically using the "Transform" menu, with the endpoints of the segment defining the translation vector. Rotate this new point left and right by, say, 8 and 10 degrees. Iterate on the new top and bottom points created (which will form the endpoints of the new branches) by selecting the top point of the trunk and the new top point of the branch (again, order is important), then click on the "Loop" button. Do the same for the other branch point. Don't forget to hide the one extra point created in the construction. Click "Stop," then test out the script. A sample script is shown in Figure 14.

```
Given:
  1.  Point A
  2.  Point B
Steps:
  1.  Let [j] = Segment between Point A and Point B.
  2.  Let [B’] = Image of Point B translated by vector A->B
      (hidden).
  3.  Let [B’’] = Image of Point [B’] rotated 10 degrees
      about center Point B (hidden).
  4.  Let [B’’] = Image of Point [B’] rotated -8 degrees
      about center Point B (hidden).
  11.  Recurse on B and [B’’].
  12.  Recurse on B and [B’’’].
```

Figure 14: Simple Tree Script

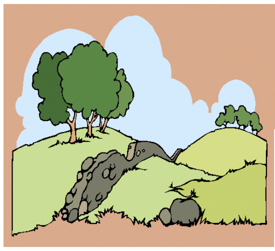
Exercises

Try a couple of variations on the above fractals. For instance, create a new circle fractal by inscribing three (or four) circles inside an initial circle, with no overlap in the circles. Make a flexible broccoli fractal where the top triangle is adjustable—that is, it is not necessarily a right triangle. Try a tree that has three branches at each joint, or four. Make a tree where each sub-branch is shorter than the originating branch by some fixed ratio. Make the angles and ratios in the tree adjustable.

Examine some fractals you have seen before, and determine how to make them as iterative structures. The Sierpinski gasket is a good place to start, as is the so-called Sierpinski carpet, which uses squares rather than triangles. The Koch Curve and Koch Snowflake are a bit more challenging, but quite do-able in GSP. Check out some fractal examples you've seen in books, and try to reproduce them in GSP. Finally, see if you can make your own new and interesting fractals.

References

A good source of information on Geometer's SketchPad is the publisher's web site at http://www.keypress.com There one can find demo versions of the software, documentation, many examples of scripts and interesting sketches, and Java implementations of the software. One can also purchase individual or classroom versions of the code at the site. There are many, many books on fractals—Mandelbrot has written very readable ones with plenty of beautiful pictures. Rather than recommending any particular book, let me suggest you see what you can find at your local library.



Math Stories

Dramatic Story of Algebraic Equations

Wieslaw Krawcewicz†

Our story begins with the familiar quadratic equation

$$x^2 + bx + c = 0.$$

This equation can be solved by applying a few simple transformations. First, by completing the square we find

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{4},$$

so, the initial equation can be reduced to

$$\left(x + \frac{b}{2}\right)^2 = \frac{b^2 - 4c}{4}.$$

If the *discriminant* $\triangle = b^2 - 4c \geq 0$, then the above equation has the solutions

$$x_1 = \frac{-b - \sqrt{\triangle}}{2} \qquad \text{and} \qquad x_2 = \frac{-b + \sqrt{\triangle}}{2}.$$

Various ways of solving quadratic equations were known to Hindu, Chinese and Arab mathematicians before they were introduced to Europe at the beginning of the Renaissance. The story related to algebraic equations took interesting turns in the sixteenth century, when many attempts were made to find similar general formulas for solving cubic and quadric equations.

A cubic equation is simply

$$x^3 + ax^2 + bx + c = 0$$

and a quadric equation is

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

The solutions of cubic and quadric equations were discovered by Italian mathematicians in the sixteenth century. The story of this discovery is dramatic and full of amazing events. The progress with solving the cubic equation was initiated by **Scipione del Ferro** (1465-1526), a professor at the University of Bologna, who solved the cubic equation $x^3 + px + q = 0$ in about 1515, but kept his work completely secret. Only just before his death did he reveal the method to his student **Antonio Fior**—a mediocre mathematician who didn't wait long to ostentatiously show off his knowledge. Prompted by rumors circulating in Bologna that the cubic equation had been solved, **Niccoló Fontana Tartaglia** managed to solve the equation of the form $x^3 + px^2 + q = 0$ and made no secret of this. Fior challenged Tartaglia to a public contest in which each of them gave 30 problems to the other with 40-50 days to solve them. Just 8 days before the contest Tartaglia found a general method to solve all types of cubic equations, while Fior was able to solve only equations of the form $x^3 + px + q = 0$. Tartaglia triumphed completely by solving all of Fior's problems within two hours.

The news about this event reached **Girolamo Cardano** (also known as Cardan) (1501-1576) who, despite being a prominent mathematician, was also a shrewd individual using his great knowledge in gambling and shady business and in practicing medicine for rich and influential people in Milan. Cardan invited Tartaglia to Milan and after much persuasion, convinced him to divulge the secret of his solution. However, Cardan made a promise in writing to keep the solution secret until Tartaglia had published it himself. Cardan broke his promise, and in 1545 he published the first Latin treatise on algebra entitled *Ars Magna*. Tartaglia protested but instead of receiving an apology, he himself was accused of plagiarism of del Ferro's work.

Cardan encouraged his most talented student, **Lodovico Ferrari** (1522-1565) to examine quadric equations. He managed to solve them in a very elegant way. The secret of solving cubic and quadric equations is based on a trick—pretending that the square roots of negative numbers, like $\sqrt{-1}$, exist and that it is possible to manipulate on these "non-existing" roots. Of course, at that time numbers like $\sqrt{-1}$ made no sense and perhaps for this reason, they were called "imaginary numbers." Nevertheless, the roots of cubic and quadric equations found in this way were completely genuine. This was the beginning of complex numbers.

In 1673, **John Wallis** presented a geometrical interpretation of complex numbers that was close to what we use today. The Swiss mathematician **Léonard Euler** (1707-1783) in 1777 proposed to use the symbol i, where i stands for *imaginary*, instead of $\sqrt{-1}$. This unfortunate choice of terminology remains to this day. Euler was the greatest man of science that Switzerland had produced. His father wanted Léonard to succeed him as a preacher in the village church, but fortunately made the mistake of

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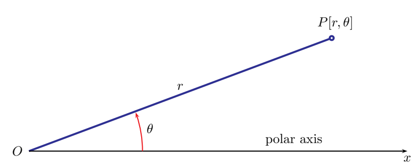
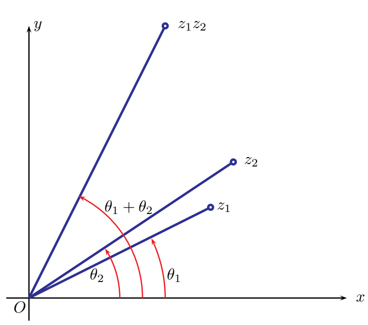
teaching him mathematics. At that time Russia, like the U.S.A. today, was providing the best career opportunities for scientists. With the backing of his friends Daniel and Nicolaus Bernoulli, who worked in the St. Petersburg Academy in Russia, Euler was able to obtain a position in the medical section of the Academy. Later, after the return of Daniel Bernoulli to Switzerland in 1733, Euler succeeded him, at the age of 26, as the leading mathematician at the Academy. He was very fond of children (he had 13 of his own) and would often compose his memoirs with a baby on his lap.

Let us return to our topic. The nature of complex numbers was not clearly understood for many years. **Caspar Wessel** in 1799 used the geometric interpretation of complex numbers as points in a plane, which made them somewhat more concrete and less mysterious, but his work received no attention. Finally in 1831, **Carl Friedrich Gauss** (1777-1855) published a paper in which he laid down a formal and correct background for complex numbers.

So, what are those mysterious *complex numbers*? Complex numbers are simply points (or vectors) in the plane for which the addition is done by taking sum of corresponding coordinates. The meaning of multiplication of complex numbers is less evident. A point P in the plane can be also represented by a pair of numbers $[r, \theta]$ called *polar coordinates* of P, where r is the distance of P from the origin $O = (0, 0)$ and θ is the angle between the x-axis and the line OP. There is a convention that the angle is assumed to be positive if it is measured in the counterclockwise direction from the polar axis, and it is negative in the clockwise direction.



Suppose z_1 and z_2 are two complex numbers (points in the plane) with polar coordinates $[r_1, \theta_1]$ and $[r_2, \theta_2]$ respectively. Then the product z_1z_2 is the point with polar coordinates $[r_1r_2, \theta_1 + \theta_2]$:



Every complex number $z = (x, y)$ can be written as $z = x(1, 0) + y(0, 1)$, where $(1, 0)$ represents the real number 1. If we denote $i = (0, 1)$, then z can be written as $x + iy$. Notice that $i^2 = -1$, thus i is indeed a root of the equation $x^2 + 1 = 0$, but the notation $\sqrt{-1}$ is wrong and leads to fallacies like $1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1} \cdot \sqrt{-1} = -1$. The form $z = a + ib$ of a complex number, which is called standard, was introduced by Carl Friedrich Gauss, who among other things, also proved the *Fundamental Theorem of Algebra*, which states that every algebraic equation

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x^1 + a_n = 0, \qquad n \geq 1,$$

where a_1, a_2, ... , a_n are real or even complex numbers, always has a complex root. With the proper foundation, complex numbers were finally accepted by the mathematical community.

Finding further formulas for solving quintic (of degree 5) and higher equations was a dominating problem in mathematics of the seventeenth and eighteenth centuries, but in spite of enormous work, a general formula for solving equations of the fifth degree

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

was not found.

Finally, young Norwegian mathematician **Niels Henrik Abel** (1802-1829), at the age of 23, proved that it is impossible to find such a formula for a general equation of the fifth degree. Surprisingly, this great discovery didn't bring much appreciation to its author in the mathematical community. Abel himself had to pay for the printing of his manuscript. Gauss, after receiving Abel's manuscript, tossed it aside with a discrediting annotation "*Here is another of those monstrosities!*" Abel received no better reception from other leading mathematicians of his time, among them Legendre, Cauchy, Hachette, etc. He was unable to get an academic position at a university and was forced to accept substitute-teaching positions. Deeply in debt, Abel died of tuberculosis at the age of 27.

The problem of algebraic equations was finally completely solved by a young French genius, **Évariste Galois** (1811-1832). Unfortunately, the circumstances of Galois' discovery were more tragic than the misfortune of Abel. At the age of 16, Galois was already well started in the research that led to his fundamental discovery. At that time he was still a high school student, but you would be wrong to assume that his incredible mathematical talent was recognized or even noticed by his teachers. His literature teacher wrote about him: *"This is the only student who has answered me poorly; he knows absolutely nothing ... I believed him to have but little intelligence. He succeeded in hiding such as he had from me."* At the age of 17, Galois sent his fundamental discoveries to Cauchy, who simply "forgot" to present his manuscript to the Academy of Sciences. Because of the ignorance and stupidity of his examiners, he twice failed the math entrance exam to the Polytechnique in Paris. As a consequence, his career possibilities to become a professional mathematician were lost forever. He was admitted to the less prestigious École Normale Supérieure and turned to political activism. A second memoir, which he submitted in 1830 to the Academy of Sciences, was lost by Jean-Baptiste-Joseph Fourier. When Galois wrote a vigorous article expressing his political views, he was promptly expelled from the École Normale Supérieure and subsequently arrested twice for republican activities. His third memoir in 1831 was returned by Siméon-Denis Poisson with a note that it was virtually incomprehensible and should be expanded and clarified. Galois died at the age of 21 in a duel, which was probably plotted by secret police. Anticipating his death, in the last hours before the dawn in the coming duel, Galois in feverish haste wrote a scientific testament in which he expressed the ideas that have kept mathematicians busy for hundreds of years. The last words scribbled by Galois were, *"I have not time; I have not time."* He was buried in a common ditch. His work was published 50 years after his death, a total of 60 pages that counts as one of the most significant achievements of the nineteenth century—the *Galois Theory*.

Math Jokes

A group of Polish scientists decided to flee their repressive communist government (it probably happened before the year 1990) by hijacking an airliner and forcing the pilot to fly them to a western country. They drove to the airport, forced their way on board a large passenger jet, and found there was no pilot on board. Terrified, they listened as the sirens got louder. Finally, one of the scientists suggested that since he was an experimentalist, he would try to fly the aircraft.
He sat down at the controls and tried to figure them out. The sirens got louder and louder. Armed men surrounded the jet. The would be pilot's friends cried out, "*Please, please take off now!!! Hurry!!!*"
The experimentalist calmly replied, "*Have patience. I'm just a simple Pole in a complex plane.*"

A mathematician wandered home at 3 a.m. His wife became very upset, telling him, "*You're late! You said you'd be home by 11:45!*" The mathematician replied, "*I'm right on time. I said I'd be home by a quarter of twelve.*"

A quiet little man was brought before a judge. The judge looked down at the man and then at the charges, then back at the little man in amazement. "*Can you tell me in your own words what happened?*" he asked the man.

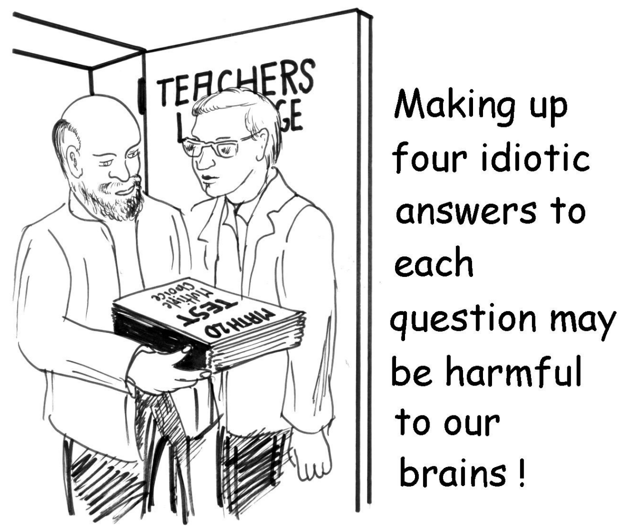
"I'm a mathematical logician dealing in the nature of proof." "Yes, go on," said the astounded judge.

"Well, I was at the library and I found the books I wanted and went to take them out. They told me my library card had expired and I had to get a new one. So I went to the registration office and got in another line. And filled out my forms for another card. And got back in line for my card."

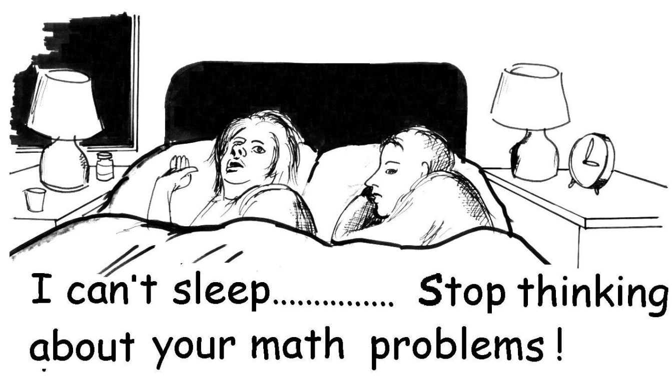
"And?" said the judge.

"And he asked 'Can you prove you are from New York City?' ... So I **punched** *him."*

Life is complex. It has real and imaginary components.
What keeps a square from moving? Square roots, of course.
The law of the excluded middle either rules or does not rule.
I heard that parallel lines actually do meet, but they are very discrete.
In modern mathematics, algebra has become so important that numbers will soon have only symbolic meaning.
Some say the Pope is the greatest cardinal. But others insist this cannot be so, as every pope has a successor.



It is true that Johannes Kepler had an uphill struggle in explaining his theory of elliptical orbits to the other astronomers of his time. And it is also true that his first attempt was a failure. But it is not true that after his lecture the first three questions he was asked were: *"What is elliptical?" "What is an orbit?"* and *"What is a planet?"*



Mathematics is made up of 50 percent formulas, 50 percent proofs and 50 percent imagination.

It is true that August Möbius was a difficult and opinionated man. But he was not so rigid that he could only see one side to every question.

Have You Used Illegal Drugs Lately? or How to Ask Sensitive Questions

Carl Schwarz†

It is often important to estimate activities that are not easily measured. For example, if we are trying to find out what fraction of youth under 20 used recreational drugs in the last year or what fraction of people cheat on their taxes, a simple telephone survey is not likely to yield useful information!

Randomized response surveys are a type of survey that is used by statisticians when asking sensitive questions. These surveys maintain the confidentiality of the responses. The respondent first uses a RANDOMIZATION DEVICE, such as a die, to select one of two questions to answer (for example by checking a YES or NO box that follows the questions). The respondent does not show the interviewer the die. For example, the respondent rolls a die. If the numbers 1, 2, or 3 are on top, the respondent answers the question:

- My mother's birthday is in January to June.

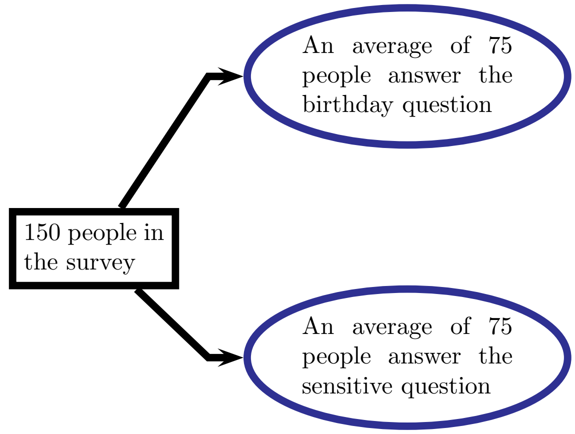
If the numbers 4, 5, or 6 are on top, the respondent answers the sensitive question:

- I have cheated on my income tax form.

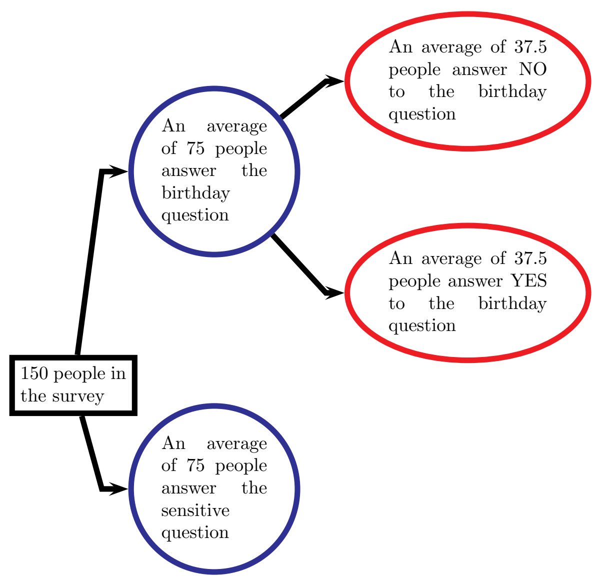
Consequently, the interviewer **DOES NOT KNOW** why a respondent answered **yes** or **no**. The confidentiality of the respondent is guaranteed.

By applying the laws of probability, it is possible to estimate the proportion of people who said yes to the sensitive question. Suppose that we interview 150 people of whom 68 said YES (we do not know at this point how many said YES to the birthday question and how many said yes to the sensitive question).

1. Because the numbers 1, 2, or 3 will come up on the die an average of 1 in 2 times, the AVERAGE number of people who answer the birthday question should be $\frac{1}{2} \times 150 = 75$ people.

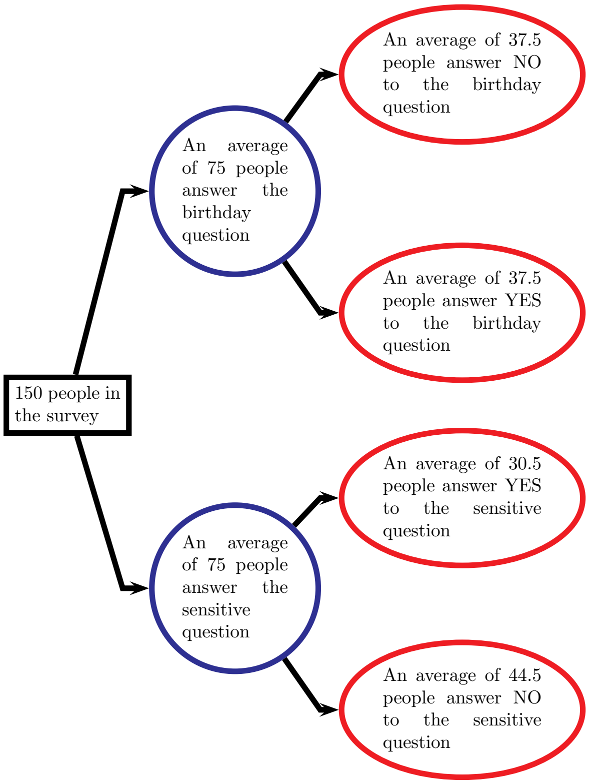


2. Because January through June is one-half of the year, about one half of the birthdays should occur during those months. Consequently, the AVERAGE number of people who say YES to the birthday question should be one-half of the average number of people who answer the birthday question, or: $\frac{1}{2} \times \frac{1}{2} \times 150 = 37.5$ people.



3. Because we have a total of 68 people who said yes to either question, ON AVERAGE, there must have been: $68 - 37.5 = 30.5$ people, on average, who said yes to the sensitive question.

†**Carl Schwarz** is a professor at Simon Fraser University in the Department of Statistics & Actuarial Science. He can be reached *via* his web page: http://www.math.sfu.ca/~cschwarz/.



4. Lastly, we estimate the proportion of people who said yes to the sensitive question as:

$$\text{estimated proportion} = \frac{\text{average number saying yes to the sensitive question}}{\text{average number answering the sensitive question}}$$

$$= \frac{30.5}{75} = .407 \quad \text{or} \quad \text{about } 41\%.$$

It is also possible to estimate the precision of this estimate (the margin of error).

Further Reading: Fox, J.A. and Tracy, P.E. (1986). *Randomized Response: A method for sensitive surveys.* Sage University Paper series on Quantitative Applications in the Social Sciences, 58. Beverly Hills: Sage Publications.

You can send your comments or questions about this article directly to the author by E-mail at cschwarz@cs.sfu.ca.

Math Jokes

HOW TO PROVE IT:

Proof by example:

- The author gives only the case $n = 2$ and suggests that it contains most of the ideas of the general proof.

Proof by intimidation:

- "*Trivial.*"

Proof by vigorous hand waving:

- Works well in a classroom or seminar setting.

Proof by cumbersome notation:

- Best done with access to at least four alphabets and special symbols.

Proof by exhaustion:

- An issue or two of a journal devoted to your proof is useful.

Proof by omission:

- "*The reader may easily supply the details,*"
- "*The other 253 cases are analogous . . .* "

Proof by obfuscation:

- A long plotless sequence of true and/or meaningless syntactically related statements.

Dana Angluin, Sigact News, Winter–Spring 1983, Volume 15 #1

There are three kinds of mathematicians: those who can count and those who can't.

There are two groups of people in the world: those who believe the world can be divided into two groups of people, and those who don't.

There are two groups of people in the world: Those who can be categorized into one of two groups of people, and those who can't.

A mathematician, a biologist and a physicist are sitting in a street cafe watching people going in and coming out of the house on the other side of the street.

First they see two people going into the house. Time passes. After awhile they notice three people coming out of the house.

The physicist: "*The measurement wasn't accurate.*"

The biologist: "*They have reproduced.*"

The mathematician: "*If now exactly one person enters the house then it will be empty again.*"

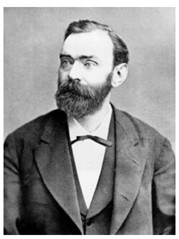
Math Excellence

The Top Mathematics Award

Florin Diacu[†]

Fields Medal

There is no Nobel Prize for mathematics. Its top award, the Fields Medal, bears the name of a Canadian.



Alfred Nobel

In 1896, the Swedish inventor Alfred Nobel died rich and famous. His will provided for the establishment of a prize fund. Starting in 1901 the annual interest was awarded yearly for the most important contributions to physics, chemistry, physiology or medicine, literature, and peace. The economics prize appeared later founded by the Central Bank of Sweden in 1968 to commemorate its 300th anniversary.

Why did Nobel choose these fields? Nobel, the inventor of dynamite, loved chemistry and physics. Literature was his great passion; in spite of a busy life, he found time to read and write fiction. Medicine and peace were natural choices for the benefit of humankind. But what about mathematics?

Gösta Mittag-Leffler

Rumour has it Gösta Mittag-Leffler, a charismatic professor at the University of Stockholm, had an affair with Nobel's wife. Outraged at discovering the liaison, Nobel damned all mathematicians. The gossip, however, is groundless; Nobel never married.

Still, a kernel of truth exists. During the decade he spent in Europe, Canadian mathematician John Charles Fields developed a close friendship with Mittag-Leffler. A colleague of Fields at the University of Toronto, J.L. Synge, recalled in 1933, "I should insert here something that Fields told me and which I later verified in Sweden, namely, that Nobel hated the mathematician Mittag-Leffler and that mathematics would not be one of the domains in which the Nobel prizes would be available."

John Charles Fields

Whatever the reason, Nobel had little esteem for mathematics. He was a practical man who ignored basic research. He never understood its importance and long term consequences. But Fields did, and he meant to do his best to promote it.

Fields was born in Hamilton, Ontario in 1863. At the age of 21, he graduated from the University of Toronto with a B.A. in mathematics. Three years later, he finished his Ph.D. at Johns Hopkins University and was then appointed professor at Allegheny College in Pennsylvania, where he taught from 1889 to 1892. But soon his dream of pursuing research faded away. North America was not ready to fund novel ideas in science. Then, an opportunity to leave for Europe arose.

For the next 10 years, Fields studied in Paris and Berlin with some of the best mathematicians of his time. After feeling accomplished, he returned home—his country needed him. In 1902, he received a special lectureship at the University of Toronto and in 1923, he was promoted to research professor, a position he kept for life. He was also elected Fellow of the Royal Societies of Canada in 1907 and London in 1913.

As organizer and president of the 1924 International Congress of Mathematicians (ICM) in Toronto, Fields attracted many sponsors and saved a large amount of money. The Committee he chaired decided to use this fund for establishing an outstanding award. Against the nationalistic mood of his time, Fields proposed that the prize be "as purely international and impersonal as possible" and that the name of no country, institution, or person be attached to it.

In the following years, he continued to lobby the international acceptance of this idea. At the beginning of 1932, the Committee's proposal was submitted to the ICM, to be held in September in Zürich. But in May, Fields fell seriously ill and sensed his end approaching. With Synge as a witness, he dictated his will. His estate was to be donated for the establishment of the prize. On August 9, Fields died of a severe stroke.

One month later, the ICM adopted the proposal with an overwhelming majority. To respect Fields' wish, the award was named the "International Medal for Outstanding Discoveries in Mathematics." But everybody called it the "Fields Medal." At the ICM in 1936 in Oslo, the first two prizes were awarded to a Finn, Lars Ahlfors, and an American, Jesse Douglas.

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In agreement with Fields' proposal that the prize recognize both existing work and the promise of future achievement, eligibility is restricted to mathematicians under the age of 40. Four awards are now given every four years at the opening of the ICM. Each consists of a medal and $15,000 Cdn, a modest sum compared to the Nobel Prize.

The medal, struck by the Royal Canadian Mint, is a gold plated cast, 25 centimeters in diameter. Designed in 1932 by the Canadian sculptor Robert Tait McKenzie, it shows the profile of Archimedes and a Latin quotation attributed to him: TRANSIRE SUUM PECTUS MUNDOQUE POTIRI (to rise above human limitations and grasp the world). The reverse side bears the inscription: CONGREGATI EX TOTO ORBE MATHEMATICI OB SCRIPTA INSIGNIA TRIBUERE (mathematicians from all over the world gathered here to honour outstanding achievement).

McKenzie had his own impression about the greatest mathematician of antiquity. In 1932 he wrote to Synge: "I feel a certain amount of complacency in having at last given to the mathematical world a version of Archimedes that is not decrepit, bald-headed, and myopic, but which has the fine presence and assured bearing of the man who defied the power of Rome." Since 1936, 42 mathematicians have received the Fields Medal. Their names and affiliations at the time of the award are provided in the table below. The country indicates the location of the institution, not the nationality of the recipient. The first Fields Medals of the 21st century will be awarded in the year 2002 in China.

1936	Lars V. Ahlfors	Harvard University, USA
	Jesse Douglas	M. I. T., USA
	Fields Medals were not	awarded during WW II
1950	Laurent Schwartz	University of Nancy, France
	Alte Selberg	Institut des Hautes Études
1954	Kunihiko Kodaira	Princeton University, USA
	Jean-Pierre Serre	University of Paris, France
1958	Klaus F. Roth	University of London, UK
	René Thom	University of Strasbourg, France
1962	Lars V. Hörmander	University of Stockholm, Sweden
	John W. Milnor	Princeton University, USA
1966	Michael F. Atiyah	Oxford University, UK
	Paul J. Cohen	Stanford University, US)
	Alexander Grothendieck	University of Paris, France
	Stephen Smale	Univ. California, Berkeley, USA
1970	Alan Baker	Cambridge University, UK
	Heisuke Hironaka	Harvard University, USA
	Serge P. Novikov	Moscow University, USSR
	John G. Thompson	Cambridge University, UK
1974	Enrico Bombieri	University of Pisa, Italy
	David B. Mumford	Harvard University, USA
1978	Pierre R. Deligne	Institut des Hautes Études Scientifiques, France)
	Charles L. Fefferman	Princeton University, USA
	Gregori A. Margulis	Moscow University, USSR
	Daniel G. Quillen	M.I.T., USA
1982	Alain Connes	Institut des Hautes Études Scientifiques, France
	William P. Thurston	Princeton University, USA
	Shing-Tung Yau	Institute for Advanced Study, Princeton, USA
1986	Simon Donaldson	Oxford University, UK
	Gerd Faltings	Princeton University, USA
	Michael Freedman	University of California, San Diego, USA
1990	Vladimir Drinfeld	Physical Institute Kharkov, USSR
	Vaughan Jones	University of California, Berkeley, USA
	Shigefumi Mori	Kyoto University, Japan
	Edward Witten	Institute for Advanced Study, Princeton, USA
1994	Pierre-Louis Lions	University Paris-Dauphine, France
	Jean-Christophe Yoccoz	University Paris-Sud, France
	Jean Bourgain	Institute for Advanced Study, Princeton, USA
	Efim Zelmanov	University of Wisconsin, USA
1998	Richard E. Borcherds	Cambridge University, UK
	W. Timothy Gowers	Cambridge University, UK
	Maxim Kontsevich	Institut des Hautes Études Scientifiques, France
	Curtis T. McMullen	Harvard University, USA

Math Jokes

HOW TO PROVE IT:

Proof by wishful citation:
- The author cites the negation, converse, or generalization of a theorem from the literature to support his claims.

Proof by funding:
- How could three different government agencies be wrong?

Proof by eminent authority:
- "*I saw Karp in the elevator and he said it was probably NP–complete.*"

Proof by personal communication:
- "Eight-dimensional coloured cycle stripping is NP–complete [Karp, personal communication]."

Proof by reduction to the wrong problem:
- "To see that infinite-dimensional coloured cycle stripping is decidable, we reduce it to the halting problem."

Proof by reference to inaccessible literature:
- The author cites a simple corollary of a theorem to be found in a privately circulated memoir of the Slovenian Philological Society, 1883.

Proof by importance:
- A large body of useful consequences all follow from the proposition in question.

Proof by accumulated evidence:
- Long and diligent search has not revealed a counterexample.

Proof by cosmology:
- The negation of the proposition is unimaginable or meaningless. Popular for proofs of the existence of God.

Dana Angluin, Sigact News, Winter–Spring 1983, Volume 15 #1

Math Inspiration

Some Famous Physicists

Anton Z. Capri[†]

Sometimes we are influenced by teachers in ways that, although negative, lead to positive results. This happened to Werner Heisenberg[1] and Max Born,[2] both of whom started out to be mathematicians, but switched to physics due to encounters with professors.

As a young student at the University of Munich, Heisenberg wanted to attend the seminar of Professor F. von Lindemann,[3] famous for solving the ancient problem of squaring the circle. Heisenberg had read Weyl's book *Space, Time, Matter* and, both excited and disturbed by the abstract mathematical arguments, had decided to study mathematics. His father, who taught Greek at the University of Munich, arranged for him to attend an interview with the famous professor so that he could obtain permission to attend the seminar.

Werner Heisenberg

When Heisenberg entered the gloomy office, furnished in a formal, old-fashioned style, he almost immediately felt a sense of oppression. A little black dog cowered on the desk in front of the professor, who glared at him with open hostility. The young Heisenberg was so flustered that he began to stammer so that his request sounded immodest even to his own ears. The little dog must have sensed his master's irritation and began to bark loudly. The professor's attempts to calm the mutt were to no avail, and so the interview turned into a shouting match. Finally, Lindemann asked what Heisenberg had read. Heisenberg mentioned Weyl's book. Over the incessant yapping of the dog, Lindemann shouted, "*In that case you are completely lost to mathematics.*"

F. von Lindemann

So, since he was "unfit" for mathematics, physics benefitted and physicists are duly grateful to Lindemann. Heisenberg's first graduate student was Felix Bloch. One day, while walking together, they started to discuss the concepts of space and time. Bloch had just finished reading Weyl's book *Space, Time, Matter*, the same book that Heisenberg had read as a young man. Still very much under the influence of this scholarly work, Bloch declared that he now understood that space was simply the field of affine transformations. Heisenberg paused, looked at him, and replied, "*Nonsense, space is blue and birds fly through it.*"

There is another version of why Heisenberg switched to physics. At an age of 19, Heisenberg went to the University of Göttingen to hear lectures by Niels Bohr.[4] These lectures were attended by physicists and their students from various universities and were jokingly referred to as "*the Bohr festival season.*" Here, Bohr expounded on his latest theories of atomic structure. The young Heisenberg in the audience did not hesitate to ask questions when Bohr's explanations were less than clear. This so impressed Bohr that after the lecture he invited the young man to go for a walk with him, drink beer, eat a snack, talk about physics, and "*have a good time.*" The excursion, which lasted several hours, impressed Bohr with the young man's talents. In turn, Heisenberg was impressed with the Danish physicist's way of attacking problems by trying to match ideas with experimental results before attempting a deep mathematical analysis. Also, Bohr acknowledged that he did not know the answers to many of Heisenberg's questions, making the problems come alive to the young man.

Niels Bohr

An amusing sequel occurred the following evening. While at a banquet, two German policemen in uniform came to "*arrest*" Bohr for "*kidnapping small children.*" The policemen were two graduate students playing a prank.

Born also started out to be a mathematician but his relations with F. Klein were not good. Klein's lectures were too polished, so Born skipped them and had a classmate keep him informed. Due to this classmate's illness Born learned with only short notice that he was to give a report on a problem in elasticity. Since he did not have time to study the literature, he developed his own ideas. This impressed Klein, so he suggested the problem for the annual university prize and wrote to Born that he expected him to submit a paper. Although reluctant, Born submitted a paper and won. Thereafter, Born switched from mathematics to astronomy in order not to have to be examined

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[1]**Werner Heisenberg** (1901-1976)—German physicist and philosopher who discovered a way to formulate quantum mechanics in terms of matrices (1925). For that discovery, he was awarded the 1932 Nobel Prize for Physics. In 1927 he published his indeterminacy, or uncertainty, principle.

[2]**Max Born** (1882-1970)—German physicist, winner of the Nobel Prize for Physics in 1954, with Walther Bothe of Germany, for his statistical formulation of the behaviour of subatomic particles.

[3]**Ferdinand von Lindemann** (1852-1939)—German mathematician who was the first to prove that π is transcendental, that is, π is not a root of any algebraic equation with rational coefficients.

[4]**Niels Bohr** (1885-1962)—Danish physicist who made numerous contributions to our understanding of atomic structure and quantum mechanics. He won the 1922 Nobel Prize for physics, for his work on the structure of atoms.

by Klein, in whose bad graces he remained. So again, physics gained and mathematics lost.

Max Born

On his final exam in 1906, Born was asked by Schwarzschild in the astronomy section, "*What would you do if you saw a falling star?*" Born confidently replied, "*I'd make a wish.*" David Hilbert[5] laughed, but Schwarzschild repeated, "*So you would, would you? But, what else?*" By now Born had collected his thoughts and explained in great detail all the observations he would make.

In 1907, when Born was an advanced student in Cambridge, he took an advanced course in electricity. A pretty young lady from Newsham, who seemed shy and standoffish to Born, who was also shy, was taking the course with him. The instructor was an old professor with an impish sense of humour, Dr. Searle. One day, when Born and his lab partner were having difficulty with their equipment, Born asked Searle for some help, "*Dr. Searle, something is wrong here. What shall I do with this angel?*" Of course he meant "*angle*." Old Searle peered at both of them over his spectacles, wagged his head and said, "*Kiss her, man, kiss her*." After that Born's shyness was even greater.

The following story illustrates the difference between scientific and military thinking. During WWI, two Swedish inventors were sent to Berlin by the German ambassador in Stockholm. These two gentlemen were lodged in the Adler, the best hotel in Berlin, with all expenses covered. Clearly, they were not too anxious to leave. Ostensibly, they had invented an ammunition detector. It was the job of a military scientist, namely Born, to evaluate their discovery.

At first the two gentlemen claimed that they needed ten kilograms of platinum. After much discussion, this was reduced to a more reasonable amount. Next they refused to have their instrument tested anywhere except at the front where, as Born repeatedly pointed out, no controlled experiment was possible. Thus, weeks went by with these two enjoying the hospitality of the Adler. Finally, they agreed to tests at an experimental station.

A hundred boxes, of which two contained ammunition and 98 contained sand, were arranged in a circle. Of course, many high-ranking officers were present. After a long and mysterious preparation, the men began to turn their instrument and, sure enough, found one of the boxes of ammo. The general shouted, "*Well done!*" and to the scientists, among whom was Born, "*Your skepticism is for once disproved. I think the test is over.*"

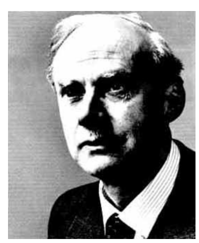
[5]**David Hilbert** (1862-1943)—famous German mathematician who contributed substantially to the establishment of the formalistic foundations of mathematics.

It took much effort by Born to persuade the general to repeat the test. This time the test failed, as it did in all subsequent cases. In the end, to show that no swindle had occurred, Born had to have all 100 boxes opened to show that two contained ammunition. By that time the general had disappeared.

When Born was already famous, he gave a lecture at the Cavendish Laboratory. Rutherford, who had little use for theorists and ruled the Cavendish with his booming voice and huge frame, made the following statement to him. "*There must be an experimental physicist with exactly the same name as you. Because, when I was preparing a lecture on the kinetic theory of gases I found a paper in the Physikalische Zeitschrift signed by Max Born and Elisabeth Bormann, and it contained the description of an experiment much too good to have been written by a mathematician like you.*" Actually there was only one Max Born and the experiment referred to dealt with the measurement of molecular cross-sections by measuring the intensity of a beam of silver atoms in vacuum and in gas.

After the Nazis came to power, Max Born left his native Germany and accepted an invitation to Cambridge. On his arrival at the Cambridge railway station, he was severely shocked to see a gigantic poster proclaiming *BORN TO BE HANGED*. The people from Cambridge calmed him, however, by explaining that this was only an advertisement of a play about someone born to be hanged.

Of course, sometimes things work in completely opposite ways for different people. Paul A. M. Dirac[6] was a most worthy successor to Newton's chair in Cambridge. Harish-Chandra became Dirac's assistant there. One day, while on a walk with Dirac and Nicholas Kemmer, he declared, "*I am leaving physics for mathematics. I find physics messy, unrigorous, elusive.*" To this Dirac replied, "*I am leaving mathematics for physics for the same reasons.*" They both did as they said and later were reunited at the Princeton Institute for Advanced Studies, where Harish-Chandra achieved fame as a mathematician and Dirac as a physicist.



Paul A. M. Dirac

Incidentally, Dirac abandoned any idea of doing experimental physics after an unfortunate mishap. He was trying to set up the equipment for the Millikan oil drop experiment to measure the charge of an electron. In so doing, he inadvertently made a bad connection and the high voltage knocked him out. After he revived, he lost all interest in doing further experimental work and stuck to theory.

[6]**Paul A. M. Dirac**—English physicist who predicted the existence of the "anti-particle." Dirac's contribution to physics was honored with a Nobel Prize in 1933.

Dirac was also one of the inventors of quantum mechanics in a form somewhat different from the earlier Heisenberg, Born, and Jordan version of matrix mechanics and Schrödinger's version of wave mechanics. In a series of brilliant papers, Dirac built an alternate mathematical framework that was far more appealing to physicists. These papers were so important in giving physicists a practical computational tool that when Dirac published a book on the subject in 1930, another prominent physicist, Lennard-Jones, remarked, "*An eminent European physicist, who is fortunate enough to possess a bound set of reprints of Dr. Dirac's original papers, has been heard to refer to them affectionately as his 'bible'. Those not so fortunate have now, at any rate, an opportunity of acquiring a copy of the authorized version.*"

Later, when both Heisenberg and Dirac had achieved fame, they set out together on a trip around the world. There are several amusing incidents that occurred.

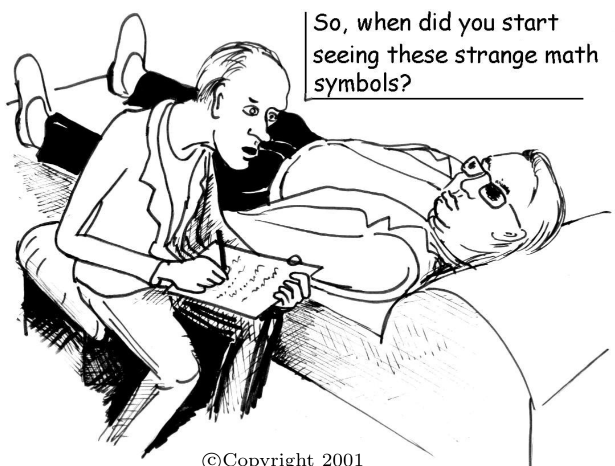
They travelled from America to the Far East on the Shinyo Maru and then later met in America again. On the first leg of this trip from the U.S. to Japan, they agreed to meet in Yellowstone National Park so that they could see some of the geysers go off. When Dirac showed up he had a detailed timetable of all the geysers that were accessible and the times at which they went off. Furthermore, he had a table of all the distances between the geysers. Using these data he had worked out a route so that it was possible for him and Heisenberg to see almost all of them go off and not waste a minute.

Heisenberg was a very active and charming man, while Dirac was somewhat shy and taciturn. While at sea, Heisenberg danced at almost every dance that was held, but Dirac sat by himself. Once, during a break in the dancing, Heisenberg returned to the table. Dirac, who had been watching all the activities, turned to Heisenberg and asked, "*Tell me, why do you dance so much?*" The ever gallant Heisenberg replied, "*When I see a nice young lady I feel compelled to dance.*" After a pause, Dirac again asked, "*Oh, but how do you know she is nice before you dance with her?*"

During their trip, the boat docked in Hawaii. Since they were due to remain there for a few days, they decided to visit the University and offer to give a seminar. After arriving at the physics department and identifying themselves, they declared their intention. Much to their surprise, they were refused. A day later another visitor from the Shinyo Maru visited the physics department. The chairman related to him with glee how the previous day two clowns claiming to be Heisenberg and Dirac had offered to give a seminar, but he had, of course, seen through them.

Dirac always abhorred being interviewed by reporters and sometimes went to extraordinary lengths to avoid them. When the boat carrying Heisenberg and Dirac docked in Japan, reporters swarmed on board to interview these two famous men. Dirac, who was standing beside Heisenberg at the railing, turned his back and stepped back. A reporter asked Heisenberg, "*Where's Dirac?*" Heisenberg simply shrugged and said nothing. The reporters interviewed Heisenberg and left. Dirac was very proud of having outwitted the reporters.

Math Jokes



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Wieslaw Krawcewicz

HOW TO PROVE IT:

Proof by picture:

- A more convincing form of proof by example. Combines well with proof by omission.

Proof by vehement assertion:

- It is useful to have some kind of authority relation to the audience.

Proof by ghost reference:

- Nothing even remotely resembling the cited theorem appears in the reference given.

Proof by forward reference:

- Reference is usually to a forthcoming paper of the author, which is often not as forthcoming as at first.

Proof by semantic shift:

- Some of the standard but inconvenient definitions are changed for the statement of the result.

Proof by appeal to intuition:

- Cloud-shaped drawings frequently help here.

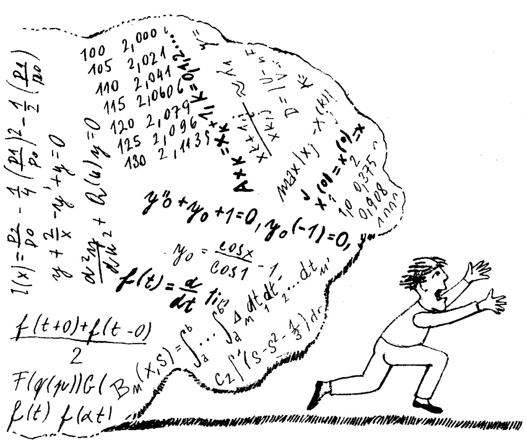
Math Games

The Chancellor of the Exchequer

Klaus Hoechsmann†

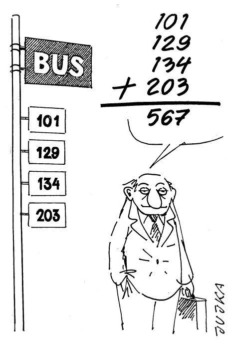
To understand why the British call their Finance Minister the "Chancellor of the Exchequer," one has to keep in mind that, from very early times (before 500 BC) until quite recently, large financial computations were done on tables marked with a kind of checker-board pattern. To calculate on such an "exchequer," tokens were moved around like beads on an abacus, but with more freedom and efficiency. In fact, the abacus was kind of a laptop version of these tables. Of course, clerks found all sorts of games to play on the desk-top model when the boss was not looking.

One of these—the game of chess—was so fascinating and time-consuming that Persia's Chancellor of the Exchequer got wind of it. Instead of risking a scandal by sacking his entire staff, he showed the game to the Shah, presenting it as his own invention. His Majesty was delighted and asked the Chancellor to name his reward. According to legend, the wily bureaucrat asked for *"just some grains of wheat"* (for details see Exercise 3); but his request was designed to humiliate the Minister of Agriculture, whose daughter was receiving more favours at Court than his own.



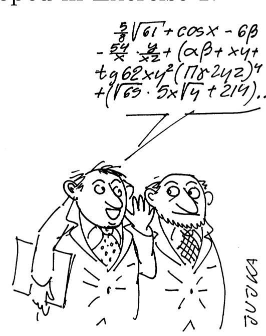
Exercises:

1. Everyone knows that a million is a thousand thousand, but there is disagreement about a billion: on this side of the Atlantic it means a thousand million (1 000 000 000), but in Europe it means a million million (1 000 000 000 000). Let us stay on this side and ask: how far is a billion millimeters, how big is a billion milliliters, how long is a billion seconds? Try to find striking ways of visualizing the Canadian federal debt of about 600 billion dollars.



2. John and Mary start pestering their parents for a Boxing Day Bonus on October 3, exactly 12 weeks before the event. Instead, Ann and Bill decide to give their kids an extra weekly allowance (payable every Friday): John starts with one dollar (on October 10) and gets a raise of \$1 every week; Mary starts with one penny and has her allowance doubled every week. How much do they have, respectively, on Boxing Day?

3. The Chancellor of the Exchequer asked for 1 grain of wheat on the first square, 2 on the second, 4 on the third, 8 on the fourth, 16 on the fifth—and so on through all 64 squares. Using the same reasoning as in Exercise 2, you will find that he would get very nearly 2^{64} grains. Is that a lot? Noting that $2^{10} = 1024$ is just over a thousand (this is the famous "K" of computers), estimate $2^{20} = 2^{10} \times 2^{10}$ (the so-called "Meg") and $2^{30} = 2^{10} \times 2^{20}$ (the so-called "Gig"). Roughly how many billion is he asking for? Try to visualize the amount using the strategies you developed in Exercise 1.



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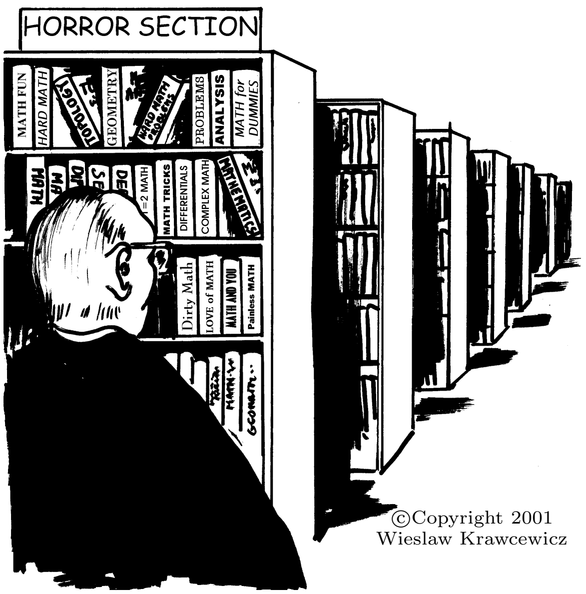
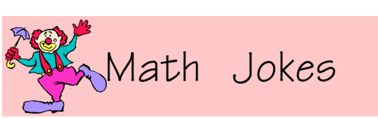
4. To correct the approximation of 2^{64} obtained in the last exercise, you would have to multiply it by 1.024 six times (i.e., apply compound interest at 2.4% for 6 periods). To improve your grain estimate, pretend that this is simple interest. What do you get?

5. The last digit given by a calculator is often uncertai My ten-digit scientific calculator, for instance, say that 2^{32} is an odd number, namely $4\ 294\ 967\ 29$ What is the correct last digit? What are the thre last digits of 2^{64} ?



6. Writing $2^{32} = a \times 10^5 + b$ with a and b less than 10^5, compute the exact value of 2^{64} using your pocket calculator. (You will have to add the various pieces by hand). If your display has only 8 digits, you can still do this by means of a finer break-up. Try to find the most efficient one in that case.

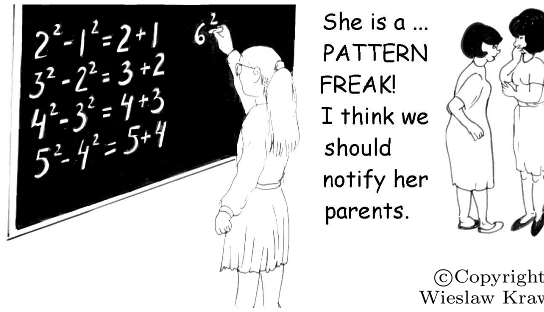
7. Continue the factorization $2^{64} - 1 = (2^{32}+1)(2^{32} - 1) = (2^{32}+1)(2^{16}+1)(2^{16}-1) = \cdots$ as far as it will go. Fermat (ca. 1650) thought that all the factors were primes. What is *your* guess? Euler (ca. 1750) found $2^{32}+1 = \left(2^{28} - 639\,(640^2+1)\right) \times 641$. He probably noticed the usefulness of $641 = 2^4 + 5^4$ and $640 = 2^7 \times 5$. How did he go from there?



There is no such place as the University of Wis-cosine, and if there was, the motto of their mathematics department would not be "Secant ye shall find."

Franklin D. Roosevelt never said "The only thing we have to sphere is sphere itself."

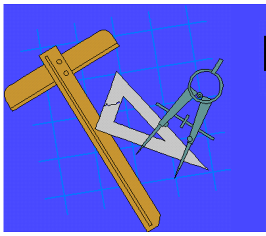
Fibonacci is not a shortened form of the Italian name that is actually spelled: *F i bb ooo nnnnn aaaaaaaa ccccccccccccc iiiiiiiiiiiiiiiiiiiii.*



How many mathematicians does it take to change a light bulb?
None. It's left to the reader as an exercise.
How many numerical analysts does it take to change a light bulb?
3.9967 (after six iterations).
How many mathematical logicians does it take to change a light bulb?
None. They can't do it, but they can easily prove that it can be done.
How many classical geometers does it take to change a light bulb?
None. You can't do it with a straight edge and a compass.
How many analysts does it take to change a light bulb?
Three. One to prove existence, one to prove uniqueness and one to derive a nonconstructive algorithm to do it.

How mathematicians do it...
Combinatorists do it as many ways as they can.
Combinatorists do it discretely.
(Logicians do it) or [not (logicians do it)].
Logicians do it by symbolic manipulation.
Algebraists do it in groups.
Algebraists do it in a ring.
Analysts do it continuously.
Real analysts do it almost everywhere.
Pure mathematicians do it rigorously.
Topologists do it openly.
Topologists do it on rubber sheets.
Mathematicians do it forever if they can do one and can do one more.
Galois did it the night before.
Möbius always does it on the same side.
Markov does it in chains.
Fermat tried to do it in the margin, but couldn't fit it in.

You Might Be a Mathematician if...
You know by heart the first fifty digits of e.
You have calculated that the World Series actually diverges.
You are sure that differential equations are a very useful tool.
When you say to a car dealer *"I'll take the red car or the blue one"* you must add *"but not both of them."*



Math Strategies

Induction Principle

Dragos Hrimiuc[†]

The Principle of Mathematical Induction is a key concept in mathematics with wide applicability. It was used in 1899 by Giuseppe Peano (1858-1932) as the fifth axiom of his axiomatic construction of the set of positive integers.

The Principle provides a technique for proving statements about positive integers, but it gives no aid in formulating such statements.

There are several equivalent formulations of the Induction Principle. We begin with the first version:

Principle of Mathematical Induction (PMI): Let $P(n)$ be a statement about the positive integer n such that:

1. $P(1)$ is true;
2. Whenever $k \geq 1$, the truth of $P(k)$ always implies that $P(k+1)$ is true.

Then $P(n)$ is true for every positive integer n.

The following is another formulation:

Principle of Mathematical Induction (modified form): Let $P(n)$ be a statement about the positive integer n and n_0 be a positive integer such that:

1. $P(n_0)$ is true;
2. Whenever $k \geq n_0$, the truth of $P(k)$ always implies that $P(k+1)$ is true.

Then $P(n)$ is true for every positive integer $n \geq n_0$.

Now, we will illustrate how this Principle can be used to solve some "non-standard" problems from various areas of mathematics.

Problem 1. Prove that $\sqrt{1+\sqrt{2+\cdots+\sqrt{n}}} < 3$ for any positive integer n.

Solution: Prove a more general inequality:

$$\sqrt{a+1+\sqrt{a+2+\cdots+\sqrt{a+n}}} < a+3 \qquad (*)$$

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for every positive integer n and $a \in [0,\infty)$. (Notice that the induction method does not work properly on the initial inequality.)

1. For $n=1$, $(*)$ transforms into $\sqrt{a+1} < a+3$, which is equivalent to
$$a+1 < (a+3)^2 \quad \Leftrightarrow \quad 0 < a^2+5a+8$$
which is valid for every $a \geq 0$.
2. Assume that $(*)$ holds for $n=k$ and every $a \in [0,\infty)$. It also holds true if a is replaced in $(*)$ by $a+1$. Hence,
$$\sqrt{a+2+\sqrt{a+3+\cdots+\sqrt{a+1+k}}} < a+4$$
or, equivalently, by adding $a+1$ to both sides
$$a+1+\sqrt{a+2+\sqrt{a+3+\cdots+\sqrt{a+1+k}}} < 2a+5,$$
that is
$$\sqrt{a+1+\sqrt{a+1+\cdots+\sqrt{a+k+1}}} < \sqrt{2a+5}.$$
From this inequality, we conclude that $(*)$ holds for $n=k+1$ if $\sqrt{2a+5} < a+3$ for every $a \geq 0$; this inequality is true since it is equivalent to $2a+5 \leq (a+3)^2$ or $0 \leq a^2+4a+4$, which is obvious. Consequently, the required inequality is satisfied for $a \geq 0$.

Problem 2. Prove that for any positive integer n there is a positive integer A of n digits, where each digit is either a 1 or a 2, such that A is divisible by 2^n.

Solution:

1. For $n=1$ we can take $A=2$, hence the statement holds for $n=1$.
2. Assume that the statement holds for $n=k$. That is, there is a positive integer $A=\overbrace{a_1\ldots a_k}$ of k digits 1 and 2, such that $2^k|A$.

Let's prove that the statement holds for $n=k+1$. Hence, we are looking for a number A' of $k+1$ digits, 1 and 2, that is divisible by 2^{k+1}. We may choose

$$A' = \begin{cases} \overbrace{2\,a_1\,a_2\,\ldots\,a_k} = 2\cdot 10^k + A & \text{if } \ 2^{k+1}|A \\ \overbrace{1\,a_1\,a_2\,\ldots\,a_k} = 10^k + A & \text{if } \ 2^{k+1}\nmid A \end{cases},$$

as the required number.

Therefore, the statement is true for $n=k+1$. By PMI, it remains true for every positive integer n.

Problem 3. A plane is divided by n lines into regions. Prove that the plane can be coloured with two colours such that all regions with a line segment as a common border have different colours (we call such a colouring a *proper colouring*).

Solutions:

1. For $n=1$ the above statement is true, since one line divides the plane into two regions that can be coloured properly.
2. Assume that the statement is valid for $n=k$, and prove it for $n=k+1$.

If $k+1$ lines are taken in a plane, any k of them divide the plane into regions that can be properly coloured (hypothesis of induction). Let ℓ denote the $(k+1)$-th line.

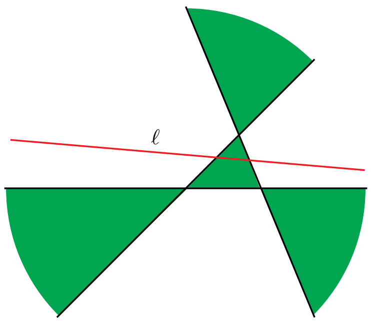


Figure 1.

This line divides the plane into two half planes, which we label Side 1 and Side 2. We leave the colours on Side 1 as they are, but on Side 2 we interchange the two colours (as shown in Figs. 1 and 2 for $k=3$).

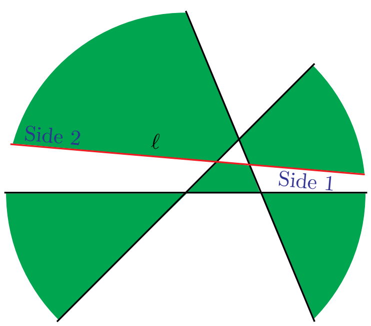


Figure 2.

The required partition is obtained. Indeed, any two regions included in Side 1 are coloured differently since they are unchanged and were assumed to be coloured differently. Any two regions included in Side 2 are coloured differently since they were initially coloured differently and we only interchanged the colours. Any two regions, one on Side 1 and the other on Side 2, that have a common border on ℓ are coloured differently. Indeed, those regions are subregions of the same part in the partition generated by the first k lines, and when ℓ was added, the colour of the subpart of Side 2 was changed to another colour, so the colours of the regions on Side 1 and on Side 2 are different.

Sometimes, the Induction Principle as initially formulated seems to be ineffective and other versions of the Induction Principle are favoured. For instance:

Principle of Complete Induction (PCI):
Let $P(n)$ be a statement about a positive integer n such that

1. $P(n)$ is true;
2. whenever $k \geq 1$ the truth of $P(1)$, $P(2)$, ..., $P(k)$ always implies that $P(k+1)$ is true.

The $P(n)$ is true for every positive integer n.

Problem 4. Let $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

(a) $f(2) = 2$;

(b) $f(n+1) = 1 + 1f(1) + 2f(2) + \ldots + nf(n)$ for every $n \in \mathbb{Z}^+$.

Solution: If $n=1$ in (b), we obtain $f(2) = 1 + 1 \cdot f(1)$ hence $f(1) = 1$. If $n = 2$, we obtain

$$f(3) = 1 + 1f(1) + 2f(2) = 1 + 1 + 4 = 6.$$

Similarly, we find $f(4) = 24$ and $f(5) = 120$. We notice a pattern:

$$f(1) = 1!, \quad f(2) = 2!, \quad f(3) = 3!, \quad f(4) = 4!, \quad f(5) = 5!.$$

Now, we prove by using PCI that $f(n) = n!$ for every $n \in \mathbb{Z}^+$. It remains to prove that if $f(1) = 1!$, ..., $f(k) = k!$ then $f(k+1) = (k+1)!$

From (b), by using the induction hypothesis, we get

$$f(k+1) = 1 + 1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k!$$

On the other hand, the identity

$$1 + 1 \cdot 1! + 2 \cdots 2! + \cdots + n \cdot n! = (n+1)!$$

holds for every $n \in \mathbb{Z}^+$ (use PMI to prove it). Hence, $f(k+1) = (k+1)!$.

Therefore, the only function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ that satisfies (a) and (b) is $f(n) = n!$.

Problem 5. Consider all subsets of the set $\{1, 2, \dots, n\}$ that do not contain any two consecutive numbers. Prove that the sum of the squares of the product of all numbers in these subsets is $(n+1)! - 1$. (Example: If $n = 3$ the subsets are $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 3\}$ and the sum is $1^2 + 2^2 + 3^2 + (1 \cdot 3)^2 = 4! - 1$).

Solution: We apply the Induction Principle on n.

1. For $n = 1$ the statement is true.
2. Assume that the statement holds for $i \leq k$ and let's prove that it remains valid for $n = k+1$. Take all subsets of $\{1, 2, \dots, k, k+1\}$ which do not contain any neighbouring elements.

We divide these subsets into two categories: (1) the subsets that contain $k+1$, and (2) the subsets that do not contain $k+1$.

By the induction hypothesis, the sum of the squares for the first category is $(k+1)^2[k!-1]+(k+1)^2$ and for the second category is $(k+1)!-1$.

Hence, the required sum is

$$(k+1)^2[k!-1]+(k+1)^2+(k+1)!-1=(k+2)!-1.$$

Therefore, the statement remains true for $n=k+1$. Consequently, by PCI it is true for every positive integer n.

Problem 6. Let x, y be nonzero real numbers such that $x+\frac{1}{x}$, $y+\frac{1}{y}$ and $xy+\frac{1}{xy}$ are integers. Prove that $x^n y^m + \frac{1}{x^n y^m}$ is an integer for any integers m and n.

Solution: It is enough to prove that $x^n y^m + \frac{1}{x^n y^m}$ is an integer for any positive integers m and n. Why?

First, we use PCI to prove that $y^m+\frac{1}{y^m}$ is an integer for any positive integer m.

1. For $m=1$ this statement is true.
2. Assume that it holds for every $m=i$, $1\le i\le k$, and let's show that it is valid for $m=k+1$.

The following identity

$$y^{k+1}+\frac{1}{y^{k+1}}=\left(y+\frac{1}{y}\right)\left(y^k+\frac{1}{y^k}\right)-\left(y^{k-1}+\frac{1}{y^{k-1}}\right)$$

can be easily verified. Since $y+\frac{1}{y}$, $y^{k-1}+\frac{1}{y^{k-1}}$, $y^k+\frac{1}{y^k}$ are integers (inductive hypothesis), we deduce that $y^{k+1}+\frac{1}{y^{k+1}}$ is an integer.

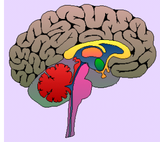
Hence, by using PCI, we see that $y^m+\frac{1}{y^m}$ is an integer for every positive integer m.

Now we prove that $xy^m+\frac{1}{xy^m}$ is an integer by using induction on m. This statement holds for $m=1$ and assuming that it is true for $m=i$, $1\le i\le k$, we show that it is valid for $m=k+1$ by using the identity:

$$\begin{aligned} xy^{k+1}+\tfrac{1}{xy^{k+1}} &= \left(y+\tfrac{1}{y}\right)\left(xy^k+\tfrac{1}{xy^k}\right)\\ &\quad -\left(xy^{k-1}+\tfrac{1}{xy^{k-1}}\right). \end{aligned}$$

Finally, we prove that $x^n y^m+\frac{1}{x^n y^m}$ is an integer by using induction on n. Indeed, the statement is true for $n=1$ and PCI can be applied by using the identity:

$$\begin{aligned} x^{k+1}y^m+\tfrac{1}{x^{k+1}y^m} &= \left(x+\tfrac{1}{x}\right)\left(x^k y^m+\tfrac{1}{x^k y^m}\right)\\ &\quad -\left(x^{k-1}y^m+\tfrac{1}{x^{k-1}y^m}\right). \end{aligned}$$



Math Challenges

Problem 1. Prove that

$$\sqrt{2\sqrt{3\sqrt{4\ldots\sqrt{n}}}}<3$$

for any positive integer n. (Hint: See the solution to Problem 1 in Math Strategies.)

Problem 2. Prove that, if $a_1, a_2, \ldots, a_n$ are positive integers, then

$$(a_1+a_2+\cdots+a_n)^2\le a_1^3+\cdots+a_n^3.$$

Problem 3. In a plane, consider n lines such that any two of them are not parallel and any three are not concurrent. Prove that the plane is divided into $1+\frac{n(n+1)}{2}$ regions.

Problem 4. Prove that n circles in a plane divide the plane into at most n^2-n+2 regions.

Problem 5. A plane is divided by n circles into regions. Show that there is a proper colouring of the plane with two colours.

Problem 6. Find $f:\mathbb{Z}^+\to\mathbb{Z}^+$ such that

(a) $f(1)=1$;

(b) $\frac{1}{f(1)f(2)}+\frac{1}{f(2)f(3)}+\cdots+\frac{1}{f(n)f(n+1)}=\frac{f(n)}{f(n)+1}$.

(Hint: For $n>1$, use the identity $\frac{1}{1\cdot 2}+\cdots+\frac{1}{(n-1)n}=\frac{n-1}{n}$ which can be proved by induction.)

Problem 7. Given n squares of arbitrary size. Prove that it is always possible to dissect the square into pieces that will form (without overlapping or holes) a bigger square.

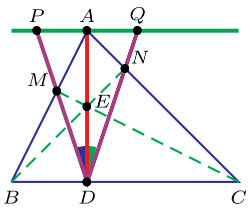
Send your solutions to π *in the Sky*: Math Challenges.

Solutions to the Problems Published in the December, 2000 Issue of π *in the Sky*:

Problem 1. (By Edward T.H. Wang from Waterloo) We may assume, without loss of generality, that $0<a\le b\le c$. Then $\frac{1}{b+c}\le\frac{1}{c+a}\le\frac{1}{a+b}$ and $bc\ge ca\ge ab$. Thus $\left(\frac{1}{b+c},\frac{1}{c+a},\frac{1}{a+b}\right)$ and (bc, ca, ab) are oppositely arranged. Hence, by the Rearrangement Inequality (see Math Strategies in the December, 2000 issue of π *in the Sky*), we have $\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\le\frac{ab}{b+c}+\frac{bc}{c+a}+\frac{ca}{a+b}$ and $\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\le\frac{ca}{b+c}+\frac{ab}{c+a}+\frac{bc}{a+b}$. Adding, we get $2\left(\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\right)\le a+b+c$ from which the result follows.

Problem 2. We will use the Box Principle (see Math Strategies in the June 2000 issue of π *in the Sky*). At any particular moment of the tournament, we can assign to each of the n players one of the boxes numbered from 0 to $n-1$, indicating the number of games played by the player. In other words, the box with number k is assigned to all players who played exactly k games. Notice that the boxes with numbers 0 and $n-1$ cannot be occupied at the same time, so we have that n players occupy at most $n-1$ boxes. Consequently, by the Box Principle, there is at least one box assigned to two players. *π in the Sky*.

Problem 3.



The parallel line through A to BC intersects DM and DN at P and respectively Q. Since $\triangle PMA \sim \triangle DMB$ and $\triangle ANQ \sim \triangle CND$, we get

$$\frac{MA}{MB} = \frac{PA}{BD}, \qquad \frac{NC}{NA} = \frac{DC}{QA},$$

and by multiplying these equalities side by side we obtain

$$\frac{MA}{MB} \cdot \frac{NC}{NA} = \frac{PA}{BD} \cdot \frac{DC}{QA}.$$

On the other hand, from Ceva's Theorem,[*]

$$\frac{MA}{MB} \cdot \frac{NC}{NA}\frac{DB}{DC} = 1 \iff \frac{MA}{MB} \cdot \frac{NC}{NA} = \frac{DC}{DB}.$$

Hence,

$$\frac{DC}{DB} = \frac{PA}{BD} \cdot \frac{DC}{QA} \iff PA = QA.$$

Therefore, $\triangle PDQ$ is isosceles (AD is the line bisector of PQ), so DA is the line bisector of $\sphericalangle MDN$.

Problem 4. We have

$$\forall_{x\in[-1,1]} \ |f(x)| \le \frac{17}{4} \iff \forall_{x\in[-1,1]} \ -\frac{17}{4} \le f(x) \le \frac{17}{4}.$$

Consequently, we have

1. $\forall_{x\in[-1,1]} \ f(x) \ge -\frac{17}{4} \iff \forall_{x\in[-1,1]} \ (x-a)^2 - 2x \ge -\frac{17}{4} \iff \forall_{x\in[-1,1]} \ (x-a)^2 \ge 2x - \frac{17}{4}$, which is true for every $a \in \mathbb{R}$, since $2x - \frac{17}{4} < 0$ for $x \in [-1,1]$.
2. $\forall_{x\in[-1,1]} \ f(x) \le \frac{17}{4} \iff \forall_{x\in[-1,1]} \ x^2 - 2(a+1)x + a^2 - \frac{17}{4} \le 0$. If we denote $g(x) = x^2 - 2(a+1)x + a^2 - \frac{17}{4}$, the above inequality is equivalent to

$$\begin{cases} \triangle > 0 \\ g(-1) \le 0 \\ g(1) \le 0 \end{cases} \iff \begin{cases} 2a + \frac{21}{4} > 0 \\ a^2 + 2a - \frac{5}{4} \le 0 \\ a^2 - 2a - \frac{21}{4} \le 0 \end{cases} \iff a \in \left[-\frac{3}{2}, \frac{1}{2}\right].$$

Problem 5. For $a, b \ge 0$ and n a positive integer, the pairs (a,b) and (a^{n-1}, b^{n-1}) are similarly arranged. Hence, by using Chebyshev's inequality (see Math Strategies in the December, 2000 issue of *π in the Sky*) we obtain

$$a^n + b^n \ge \frac{1}{2}(a+b)(a^{n-1} + b^{n-1}).$$

If we repeatedly apply this inequality, we finally get

$$a^n + b^n \ge \frac{1}{2^{n-1}}(a+b)^n.$$

Now, by taking $a = \sin^2 x$ and $b = \cos^2 x$, we obtain the required inequality. (We received another solution, using the Induction Principle, from Edward T.H. Wang from Waterloo).

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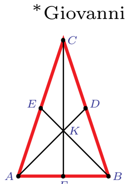
[*] Giovanni Ceva (1648-1734) proved a theorem bearing his name:



Ceva's Theorem: In a triangle ABC, three lines AD, BE and CF intersect at a single point K if and only if

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

I would like to comment on the article by A. Liu published in *π in the Sky* in June 2000. I'm a half-time janitor at the high school here in Fort MacLeod and I happened to be reading *π in the Sky* magazine in the Physics Lab. Your Math Opinion expressed in the article "*The Perfect Education System For an Affluent Society*" was, I thought, quite profound! (Not to mention ironic and funny ...).

Our society is "increasingly dominated by commercial concerns" and is, at times, one big consumer event.

Thank you once again for an astute article.

Peter Craig

Dear Editors of *π in the Sky*,

I wanted to write, first and foremost, to congratulate you on the excellent job you have done in putting together the first two issues of Pi in the Sky. It is an excellent magazine. I particularly enjoy the mix of informative articles and jokes. The article on the Game of Nim by Akbar Rhemtulla carried me back nearly 30 years to my childhood in East Africa.

I was, however, rather taken aback by a paragraph in the article by W. Krawcewicz in the December, 2000 issue:

> *"We can only speculate what would have happened if these facts were known to Columbus sixteen centuries later. However, it is very unfortunate that for almost one thousand years Western civilization was living in complete darkness, unaware of the great scientific discoveries of the ancient Greeks. This one thousand years was a great loss for mankind."*

Unless I have read the paragraph incorrectly, the author is claiming that Columbus and all of his educated contemporaries subscribed to the "flat earth hypothesis." This, in itself, is a rather controversial issue. For example, Jeffrey Burton Russell, a historian, has very persuasively argued that the "flat earth" myth was concocted and popularized by Washington Irving in the early 1800s. But even this aside, surely it seems absurd that experienced and educated navigators and sailors of the day would not be aware of the Arabs sailing down the East African coast or to the Malay archipelago, especially given that the close contact between the Spanish and Arabs on the Iberian peninsula. Or the fact that the "scientists" all across the Islamic land knew that the earth was round. Records from the Abbassid Caliphate in Baghdad, ca. 800 A.D., show that not only were the Arabs aware of Eratosthenes' measurement of the Earth's diameter but in fact, had accepted the notion fully.

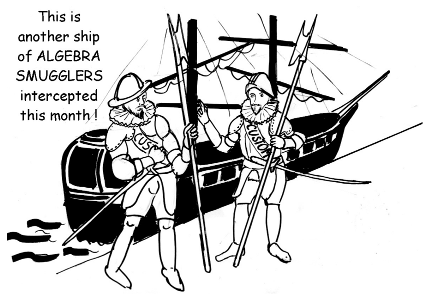
As interesting as the "flat earth" issue is, I am however much more concerned about the last sentence in the paragraph. Even if Western Europe languished in the dark for a thousand years, I am at a loss as to why that constitutes a "great loss" for mankind. I guess one can interpret that statement in two ways: first, that the decline and loss of intellectual activity—whatever that means—by any society/civilization around the world is a "great loss for mankind," in which case, I would have to agree. However, there is another interpretation that somehow Europe is special and because of the paragraph's construct, I get the impression that is indeed what the author meant.

For the record, the period between 100 A.D. and 1300 A.D. saw the rise of the Islamic empires, especially the Abbassids and the Fatimids, as well as the second, the high period, of Hindu mathematics in the Indian subcontinent. And while the scholars (Indians, Persians, Arabs, Jews, etc.) perhaps were not as devoted to the concept of proofs as the Greeks were in their geometry, the introduction of the positional notation in base 10 (using special symbols for numbers 1 to 9 and zero as a number), the introduction and use of negative numbers, the free use of irrationals as numbers, the play with indeterminate equations, the solving of equations algebraically rather than geometrically (though the latter was then used to justify the solution), to name a few, revived algebra and arithmetic, the two areas that had been subverted by the Greeks who had insisted on geometric basis for all things mathematical. One can argue that the Hindu and the Islamic work not only restored algebra and arithmetic to their proper foundation and status, placing it on par with geometry, but even advanced the art in many important respects and set the stage for their subsequent flowering.

In this day and age, I would assume that we have come to understand that the central issue is the development of ideas and knowledge. Eurocentric or Islam-o-centric or any other "centric" views are things of the past. No one civilization can lay claim of ownership to our current body of knowledge. As far as "mankind" is concerned, civilizations come and go, however, knowledge flows around and ideas had always found rich environments to take root and flourish in. And the body of knowledge that we are so proud of today bears the imprints of the philosophical approaches of all the peoples that contributed to it, imprints without which the richness and the textures that we have come to appreciate would not have arisen.

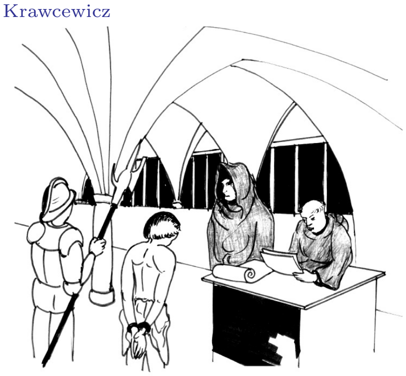
I look forward to seeing articles and stories in Pi in the Sky that capture the "international" nature of mathematics.

Sincerely, Arif Babul



I am very glad to read your comments. The passage you are referring to is touching on the issue of the mistake made by Columbus who identified the new continent as a part of Asia (India??). Knowledge of the exact circumference of the Earth would have allowed him to better estimate his position and avoid this mistake. On the other hand, if we assume that one thousand years of "dark ages" really happened (there are many reasons against it), it would indeed have been a a great loss for all of humanity.

—Wieslaw Krawcewicz



You are accused of witchcraft,
black magic and quadratic equation !

Math Links

Introducing MegaMath

The MegaMath project is intended to bring unusual and important mathematical ideas to elementary school classrooms so that young people and their teachers can think about them together:
http://www.c3.lanl.gov/mega-math/

Mathlets: Java™ Applets for Math Explorations

Contains many Java™ Applets, written by Tom Leathrum, for slopes, parabolas, periodic functions, roots of polynomials, exponential functions, conic sections, systems of linear equations, etc.
http://cs.jsu.edu/mcis/faculty/leathrum/Mathlets/

Arkadii Slinko's Mathematics Olympiad Learning Centre

This site is for those who are interested in learning mathematics and training for math competitions. Articles on elementary math written especially for high school students can be found in the Articles section.
http://matholymp.com/

Internet Learning Network

The Internet Learning Network is the web site giving students, parents and teachers a free, fast and private way to compare your science and math skills to students from around the world.
http://www.getsmarter.org/index.cfm

Platonic Realms

This site contains Math Quotes, Encyclopedia, Math Jokes and other stuff that may be interesting for students and teachers. Check it out at:
http://www.mathacademy.com/

Math2.org

The Math2.org site offers Math Reference Tables, Math Message Board, Math Forum and other interesting links.
http://www.math2.org/index.xml

S.O.S. Mathematics

S.O.S. MATHematics is a free resource for math review material from Algebra to Differential Equations.
http://www.sosmath.com/

The KnotPlot Site

Here you will find a collection of knots and links, viewed from a (mostly) mathematical perspective. Nearly all of the images were created with KnotPlot, a fairly elaborate program to visualize and manipulate mathematical knots in three and four dimensions. Download KnotPlot from
http://www.cs.ubc.ca/nest/imager/contributions/scharein/KnotPlot.html

Math Teacher Link

Professional Development Consortium for Mathematics Teachers.
http://mtl.math.uiuc.edu/